A BRIEF HISTORY OF DEDUCTIVE REASONING

BY ROBERT WATKINS

Mathematicians are indeed a curious breed. They want, as far as is possible, to know what they are talking about- and they are willing to go to extraordinary lengths to be certain. For example, two gifted mathematicians from this century, Bertrand Russell and Alfred North Whitehead, use the first 362 pages of their famous *Principia Mathematica*, to show that 1+1=2. To most people, such pedantic behavior is ludicrous, perhaps even a little surreal. But not to mathematicians. As the three volume *Principia* was being published in journals piecemeal, many mathematicians cheered the work along, With respect to the Philosophy of Mathematics, the *Principia's* objective was among the most serious undertakings ever attempted.

There was, of course, never any doubt that 1+1=2. Actually, Russell and Whitehead were not out to prove anything in particular-- their goal was, rather, to unify everything that had already been proven. The *Principia* is a formidable tomb. Few mathematicians ever read it. (The cheering mathematicians referred to above, in most cases, were only reading abstracts.) To be precise, no one ever 'reads' the *Principia*-- they decipher it. Minimum prerequisites for understanding it are a graduate course in formal logic, another in set theory, and several weeks (or months) of uninterrupted free time. It is only possible, in a short article such as this, to give the general idea of what Russell and Whitehead hoped to accomplish with their *Principia*, how they tried to do it, and why they failed. The account presented below presupposes no formal mathematical training beyond the high school level (Introductory Algebra and Geometry). Admittedly, for some readers, even this version may be difficult. But Russell and Whitehead's conception was so grand, their failure so profound, and their recovery so sublime, that any attempt to grasp it will assuredly be worth the effort.

The place to begin is with an understanding of how mathematics is produced. There are two aspects to a mathematician's work: induction and deduction. Induction is characterized by hunches and trials (and errors). It begins whenever a mathematician notices an unexpected pattern. This leads to a conjecture. More problems are devised and the conjecture is tested. If the pattern continues to be observed, naturally, confidence in the conjecture is bolstered. Yet, with most questions in mathematics, an infinite number of cases can arise. Under such circumstances, even a million trials will not serve to make a mathematician sure of a conjecture. For certainty deduction is required.

To deduce a result, what is already known must be built on incrementally, retaining certainty with each new step, until, at last, the desired conclusion is obtained. The end product of this process is known as a 'proof'. Constructing proofs is the main work of mathematicians. To formalize this procedure, mathematicians have developed what are known as formal (axiomatic) systems. These systems are networks of interrelated ideas built one upon another.

To prime a formal system's deductive engine, some basic statements must initially be accepted as true. These basic statements are called axioms and it is hoped that their truth is self-evident. An axiom of arithmetic is: when two whole numbers are added together, their sum is still a whole number'. Axioms are also, sometimes, referred to as postulates. Students of high school geometry may recognize the following statements as postulates of geometry: 'between any two distinct points there is one and only one straight line'; "when two distinct lines meet, they intersect at a single point'. Again, it is important to remember that the truth of these statements is accepted without "proof" and, as such, their number must be kept to a minimum.

Once a set of axioms has been agreed upon, they can be manipulated in accordance with the rules of logic to produce new statements. In mathematical terms, these new statements are called theorems. Once a theorem has been proven (i.e. deduced in accordance with the rules of logic) it too becomes part of the axiomatic system and can be used in future attempts to produce more theorems. By building a body of knowledge in this way, faith in the truth of the theorems is as strong as faith in the truth of the axioms (and the rules of logic)- which is to say, their truth is regarded as self-evident. What is most amazing about these formal systems is that with just a handful of axioms (and human ingenuity) a seemingly inexhaustible number of new theorems can be deduced, Mathematicians are currently producing new theorems at a rate of half a million per year. And each as true as 1+1=2.

The use of deductive proof is ancient. The first deductive proof is commonly attributed to a Greek named Thales of Miletus (c. 580 BCE), 'The Father of Philosophy'. According to extant sources, the invention of deductive proof arose from a practical problem. Thales was engaged in trading with both the Egyptians and the Babylonians. The merchants in each of these countries had a different formula for calculating the volume of the truncated base of a pyramid, a frustum-- but, their formulas gave different results. Thales reasoned (correctly, I assume) that they could not both be right. But to determine which was right, if indeed either was; and more, to convince anyone they were wrong; Thales knew that he must start with ideas that everyone accepted as true and then combine them one idea at a time, until, at last, a logically compelling formula was produced.

From Thales' time onward, the Greeks made extensive use of deductive reasoning. In the classical period, they perfected the rules of logic, began investigations into nearly every major branch of mathematics, and produced an astonishing array of useful theorems and clever proofs. Their contribution to mathematics is attested to in nearly every modern text by honorifics such as the Pythagorean Theorem, the Archimedian Principle, and the Sieve of Eratosthenes. Indeed, as this article will show, their impact is still being felt-- and sometimes in unforeseen and disturbing ways.

It was not until the end of the Greek classical period that anyone thought to systematize an entire branch of mathematics. History assigns this glorious accomplishment to one man, Euclid, a Greek at the Museum of Alexandria (Greek: *Musion*- temple of the Muses, goddesses of memory). In approximately 350 BCE, Euclid founded the Museum's School of Mathematics. He also wrote several books. One of these, a compendium of mathematical knowledge arranged in thirteen Books, *the Elements*, was a phenomenal success, unparalleled in history. It established once and for all, the model for clear thinking; it was the world's first formal axiomatic system.

The subject of most of the *Elements*' thirteen Books is geometry (although number theory is covered in Books VII, VIII, and IX). First printed in 1482, from a copy made by Theon of Alexandria (4th century CE.), it has been reproduced in at least 1,700 editions. Its effect on Western thought has been incalculable. Serving as the capstone to higher education for over 2,000 years, *the Elements* has been read by more people than any other book except the Bible. From the existence of Euclidean truths, the father of modern philosophy, Descartes, extrapolated the existence of God in that same famous *Meditations* wherein he claimed, 'I think, therefore I am.' Spinoza's magnum opus, the *Ethics* (whose complete title is: *Ethics, Demonstrated in, a Geometrical Manner* ; one of the greatest philosophical works of all time, took its entire structure from the *Elements*- complete with axioms, postulates, theorems, corollaries.

However, it was not until the late 18th century that Euclid's formal development of geometry was mimicked in other branches of mathematics. Why this didn't occur earlier is inexplicable; but 'once begun is nearly done' and modern mathematics has since rapidly evolved into various overlapping axiomatic systems each containing defined objects, basic assumptions, rules of inference, and copious results. The success of this undertaking is reflected in the 500,000 new theorems that are being generated each year. Without an axiomatic approach to mathematics, this would not be possible. This is Euclid's Legacy.

There always was, however, one small question concerning Euclid's Elements. This question is referred to in the work of an obscure 18th century scholar, entitled: *Euclid Vindicated from Every Flaw*. A flaw? In Euclid? Most people thought not. But, there was one tiny, nitpicky detail that had always bothered mathematicians (even Euclid himself). Most people could hardly see calling it a flaw; it was really more like a minor aesthetic point regarding **Euclid's 5th postulate**. This postulate, known as the parallel postulate, states that, '**through any point, not on a given line, there is one and only one line, through the point, parallel to the given line**'. The problem was that this postulate seemed too complicated- too wordy! Consequently, mathematicians, analytic creatures that they are, spent 2,000 years trying to say the same thing- but in fewer words.

To most people, such pedantic behavior seemed ludicrous, perhaps even a little surreal-- but not to mathematicians. They harkened to the same voice of Reason as Dante Alighieri acknowledged in saying, "All that is superfluous displeases God and nature. And all that displeases God and nature is evil." To know, as far as is possible, what they are talking about, the things which mathematicians accepted without proof must to be kept to a minimum. It was always suspected that Euclid's 5th postulate could be derived from the others. If so, it needed to be eliminated as an axiom and proved as a theorem. Or failing at this, at it could be replaced by something more concise, something somehow more self-evidently true-- which is not to imply that anyone doubted the truth of Euclid's parallel postulate. Experience with a compass and straight-edge had always supported Euclid's 5th postulate.

But was its truth self-evident? The author of *Euclid Vindicated* took a novel approach to this question. He attempted a proof by contradiction. By the rules of logic, if a statement is true, then its negation is false. In an axiomatic system, false statements lead to contradictions. Thus, the author of Euclid Vindicated denied the truth of the Parallel Postulate and searched through the theorems that logically followed from its negation for a contradiction. His efforts did produce many results strikingly different from those found in Euclid. Indeed, he eventually convinced himself that a contradiction had been obtained. But in fact, no contradiction was ever produced. It took 100 years to realize, but, unwittingly, *Euclid Vindicated* formed the basis for the first Non-Euclidean Geometry.

In the 19th century, several mathematicians began to deliberately develop Non-Euclidean geometries. Nicolai Lobachevski, for instance, considered the implications of no lines being parallel; and Janos Bolyai, contrary to Euclid's belief in one and only one line, postulated an infinite number of parallel lines through any given point. Whether or not these new geometries had any real-world applicability was initially of little importance. Experimentation and discovery have always been key aspects of mathematics and often a hypothetical is chased through a maze of deductive reasoning just to see where it will turn out.

As it turned out, these new geometries did have practical applications. For instance, Lobachevski's geometry, now known as spherical geometry, describes relations between points **on the surface** of a globe-- -the Earth for instance. Until a Trans-Atlantic tunnel is drilled, the shortest path from New York to London is a great arc ('straight' lines do not lie **on the surface** of a sphere—they are not 2D, they are disjoint, tangent, or secant to a sphere and require 3D). Similarly, Bolyai's geometry also describes the appearance of points in many regions of space (although this was not really appreciated until Einstein drafted his theory of relativity). Most cosmologists now believe that Bolyai's creation, hyperbolic geometry, may essentially describe the overall appearance of the Universe (although many cosmologists don't want to believe this and are currently searching for 'the missing mass' needed to straighten it out- but this is a tangent perilously close to a black hole—which we should try to avoid).

Acceptance of these non-Euclidean geometries complicated a question that philosophers had already been hotly debating: What was Truth? The Rationalists, like Descartes and Spinoza, claimed that truth could only be determined by studying an idealized thought process. Of course, Euclid was their idol. On the other hand, the Empiricsists, such as Berkely, Locke, and Hume, claimed that Truth, as the ancients had conceived it, was a fiction. For them, Knowledge was all that man could hope to obtain.

Convincingly, they argued that knowledge could only be gained by experience, not introspection. The discovery of new geometries clearly muddied these already turbulent philosophical waters. In the 200 years to follow, philosophers concocted numerous theories in attempts to rescue Truth. These theories, such as correspondence theory, coherence theory, pragmatism, and positivism, are still popular today; but the old fashioned brand of Certainty has long since faded from them.

For the most part, mathematicians, content with studying their numbers, equations, points and so on, bowed out of these debates. What they really believed was obvious; they treated the objects of their studies as 'real', and the conclusions they derived as 'true'. If confronted with prying metaphysical and epistemological questions, they would quickly admit that they were dealing with idealizations and that all their results were in the form of hypotheticals. But, Neo-Platonists to the core, they remained confident in the truths they were discovering.

Perhaps our modern technological society relies too heavily on the results derived from many branches of mathematics to insist that they are mere abstractions. Radioactive decay, satellite orbits, crop yields, life expectancies, and even tomorrow's weather are predicted using mathematical models. Unfortunately, this is the only aspect of mathematics that some people see. For these people, the disillusionment that the Non-Euclidean geometries brought with them is understandable. It was natural that statements about geometry were taken to be statements about the world. But mathematics, like any symbolic language, is more than just a mirror for the world. It is, in fact, the one universal human language- the language of science and the language of science fiction. With it the vastness of the Universe can be transcended. It functions equally well in Einstein's space-time or Lewis Carroll's *Wonderland*.

A real appreciation for mathematics only comes after the mind has been habituated to rigorous thinking. (Plato required students to study geometry for five years before entering the Academy.) At some point, it dawns on students that mathematics is as much an art as it is a science. Among mathematicians, proofs are referred to as elegant, or ingenious, or even beautiful. New proofs are often devised for things long since proven and novelty is highly prized. For instance, the variety of proofs for the Pythagorean Theorem, which states that 'the sum of the squares of the legs in a right triangle is equal to the square of the hypotenuse ($a^2 + b^2 = c^2$)', has been a subject of interest to mathematicians since antiquity. Many mathematicians are even proud collectors of the several hundred Pythagorean proofs already known. Aesthetically speaking, short proofs are generally preferred to long ones; direct proofs are generally preferred to indirect ones; and exhaustive proofs, totally lacking in finesse, are abhorred and avoided whenever possible. Every competent mathematician sees and understands the basis for these aesthetic criticisms. Deliberately faulty proofs can be humorous; and vacuous proofs side-splitting'.

(Have you heard the one about the logic professor trying to explain why anything follows from a false antecedent? When the professor asked a confused student if he was sure he didn't understand, the student said, "All I'm sure of is that IF I understand this stuff, THEN I'm the Wizard of Oz." And the professor said, "Aha, then you do understand!" and the whole class roared with laughter.)

Fortunately, throughout the philosophical upheaval of the 19th century, one branch of mathematics maintained its purity. Because of it perceived lack of 'real world' applications, Number Theory, the Queen of Mathematics, was spared the indignity of seeing any of her subjects lose faith. With World Wars raging and atomic bombs exploding, G. H. Hardy, England's leading 20th century mathematician said, "Both Gauss and lesser mathematicians may be justified in rejoicing that there is one science at any rate, and that their own, whose very remoteness from ordinary human activities should keep it clean and gentle."

For mathematicians, this one last realm of ethereal Epicurean delights remained, and would always remain---Number Theory! Even when attached to physical objects, numbers did not become less abstract. Admittedly, the concepts 'circle' and 'one' are both abstractions- examples of Plato's Perfect Forms, but, they do not have the quality of being abstract in the same degree. When a shape has circularity, it is, at least in some sense, a circle; but when a thing has oneness, it is not, in any sense, a one.

The concept of number plays a central role in every branch of mathematics. This is why Number Theory is often referred to as the Queen. But numbers have been something of a philosophical paradox for the last hundred years. Philosophers, in general, agree that to say, numbers do not 'exist', or that their properties are not 'real', is at most semantic, and probably ridiculous. (1+1=2) And yet, nevertheless, they clearly did not 'exist' as had previously been conceived, nor were their properties 'real' in the same sense as everyone had assumed before the creation of Non-Euclidean Geometries. Like all mathematical objects the properties of numbers must, most philosophers agreed, ultimately rest on some set of assumptions.

The pause which the development of non-Euclidean geometries caused philosophers, was accompanied by an explosion of new activity among mathematicians. Once mathematical 'truth' no longer rested, in any way, on the physical world, new branches of mathematics could be easily created. Axioms could be stated, definitions written, and theorems derived- without undue concern for worldly relevance. Whatever claim mathematics had to producing true statements depended solely on the precise observance of rules. Naturally, from this philosophical point of view, the technical details in formal axiomatic systems were all important.

Yet, as Lobachevski and Bolyai had demonstrated, formal systems based on different sets of rules, while contradicting each other, could remain internally consistent. Mathematics had evolved into various overlapping axiomatic systems; so, what guarantee was there that the rules of algebra didn't conflict with the rules of topology? No one was sure. Many of the most basic axioms, rules common to every branch of mathematics, the rules of number theory for example, had never been written down.

Mathematicians tacitly assumed that all the rules could be made explicit and reconciled without need for significant changes. This was the task that Russell and Whitehead undertook in writing their *Principia*. Their idea was to extract the salient assumptions from the various branches of mathematics and recast them in a few basic rules common to all. More specifically, they wanted to subsume the various branches of mathematics (but primarily its Queen, Number Theory) into a single branch known as set theory. Then, with a few accepted rules of logic (as few as possible), they could deduce (at least in theory) all of mathematics.

The program undertaken in the *Principia* was a refinement (and an elaboration) of Russell's earlier work, the *Principle of Mathematics* (1903). In the *Principles*, Russell makes a linguistic argument against Idealism (i.e. the view that the 'external' world is created by the mind). His solution to various paradoxes he uncovers is a strict equation of meaning with reference. Thus, for Russell, if a word has no referent it has no meaning. In constructing this definition of meaning, Russell's training as a mathematician came shining through. In mathematics, definitions are neither right nor wrong- they can only be useful or useless. Russell's definition of meaning was useful; it banished a paradox (known as Russell's Paradox) from the fledgling study of Set Theory and it showed Alfred North Whitehead, Russell's mentor, the path to their *Principia.*

The problem with the *Principles* was that, in requiring of every idea a distinct referent, a veritable Amalthea's Horn of imperceptible real entities was poured out into the world of thought, and Russell's hopes of neatly developing their logical relations was drowned in the deluge. Whitehead suggested an expedient, known as logical constructions, that greatly simplified Russell's original conception. Instead of requiring a distinct real object as referent for each meaningful term, Whitehead proposed that referents could be constructed, without loss of meaning, from already known objects- such as sets.

Georg Cantor began developing Set Theory in 1874. In 1910, when Russell and Whitehead co-opted it into their *Principia*, Set Theory was still in its infancy. But the study of sets had already provided a unique and powerful approach to number theory. For the first time, number theoretic principles were being given an explicit formulation. The results were astounding. For instance, Cantor had defined and extended the concept of number (actually, size or cardinality of a set) in an exceedingly simple way. His definition of number had pre-conscious roots; ancient Chaldean shepherds, no doubt, had the same conception. Suppose a shepherd wants to keep track of his sheep. As each animal leaves the cave in the morning, the shepherd places a stone in a pouch; then, as each of the sheep returns at night, he takes one stone out. If all the sheep come home, then the pouch must be empty On the other hand, as long as there are still stones in the pouch, there must be sheep not in the cave. There is a one to one correspondence between the number of sheep and the number of stones. The number of these two sets is, by Cantor's definition, equal.

This definition of equal size gives most people no problem when it is applied to sheep and stones. But Cantor applied this definition to other sets-- infinite sets, sets of number, e.g. {1,2,3,4.... } and {2,4,6,8....}. His conclusions deductively followed from this definition of equal size, but they are counter intuitive- for instance, the set {1,2,3,4....} has no more things in it than the set {100, 200, 300, 400....}. But, as if this were not startling enough, the set of all real numbers between 0 and 1 has infinitely more things in it than the set {1,2,3,4...}.

In the *Principia*, Russell and Whitehead developed a completely abstract formal system to deal with numbers and their arithmetic. Their system contained strings of symbols and rules for manipulating them. The initial strings were the system's axioms, the rules for manipulating these strings constituted the system's rules of logic, and the strings that could be produced were the system's theorems. No meaning was inherent to this system because of the abstract way in which it was developed; the technical details of the system were all that was important.

The system adopted in the *Principia* (sometimes referred to as a typographic system) was immensely detailed. However, even a pitifully impoverished example can explain how, in principle, it worked. For instance, suppose we are given the following (meaningless?) string:

a%b*c%b

And the following rules:

1) interchanging the sides of %
   In any string, wherever % appears, the symbols that appear next to it can be interchanged; e.g. f%g transforms to (denoted >>>) g%f
2) inserting and deleting like symbols adjacent to *
   In any statement, wherever * appears, if the symbols on both sides of the * are the same, then they can be eliminated; or, a symbol can be inserted next to * e.g. s%*%g >>> s*g or d*g >>> d%*%g

Then (by rule 1) a%b*c%b >>> a%b*b%c
(by rule 2) a%b*b%c >>> a%*%c
(by rule 2) a%*%c >>> a*c

Thus, according to the rules of the system, a%b*c%b >>> a*c

In and of itself, this system 'means' nothing. But, like the typographic system in the *Principia*, meaning can be assigned to it. In fact, any meanings can be assigned to the component parts as long as they remain consistent. For instance, suppose we equate the letters a, b, c, etc. with numbers; the % with addition, and the * with equality. Then rule 1) says that addition is commutative (e.g. 3+7 is the same as 7+3); and rule 2) allows unique cancellation (e.g. x+3=3+7 implies x=7). Because this system is so simple, the 'normal' meaning of these words (e.g. addition, commutativity, etc.) is not fully captured by the system. But, the typographic system of the *Principia* was much, much more elaborate (recall how long it took to show that 1+1=2). It did provided the material out of which Russell and Whitehead could supply referents via logical constructs for (?all?) mathematical entities,

In 1913, when the final volume of the *Principia* Mathematica was published, although there were still matters to be settled, the logical groundwork had finally been laid for an axiomatic development of mathematics, including number theory, As Russell put it, "Logic is the youth of mathematics, and mathematics is the maturity of logic." This was the sentiment that the *Principia* embodied. On the basis of the *Principia*, the belief grew that a formal axiomatic system (if not the *Principia*, a modest refinement thereof) would provide a means of proving every true theorem in mathematics.

But in 1931, this belief was permanently shattered by a young mathematician named Kurt Godel. Godel provided a proof that no axiomatic system could ever be complete. His proof was like a klieglight, sharply illuminating drawn features which time had stretched across the bare bones of logic. Godel's proof was contained in a paper entitled, "*On Formally Undecidable Propositions of Principia Mathematica and Related Systems*". When his proof was first published, few mathematicians were in a position to understand it. Russell and Whitehead were among the few. The basic idea of Godel's proof was simple and elegant, the details were overwhelming. The 'Undecidable Propositions' referred to in his paper's title, arose from an ancient logical paradox formulated by Epimenedes of Crete.

There is no more fundamental idea in logic than the law of the excluded middle. Aristotle states this law as, "The same attribute cannot at the same time both belong to and not belong to the same subject and in the same manner." For example: either an orange is a fruit, or an orange is not a fruit; either 1+1 is 3 or 1+1 is not 3; either God exists, or God does not exist. In the symbols of modern mathematics, we say: A or ~A (meaning 'the statement denoted by A is true, or the negation of A is true).

Thales used this rule of logic when he claimed that the Egyptians and the Babylonians could not both be right if their formulas for the frustum of a pyramid gave different results. Epimenedes paradox seems to point out a counter contention. Epimenedes claimed, "All Cretans are Liars" (meaning that everything they say is a lie). If his statement is true, then he must be lying, and if he's lying, then what he says must be false, but if its false, then what he says...

This paradox is widely known as the Liar's Paradox. (Even the apostle Paul had heard of Epimenedes' claim- although he may have missed the point-- Titus 1:12, "Even one of their own prophets has said, 'Cretans are always liars, evil brutes, lazy gluttons.' This testimony is true." Another formulation of this paradox, one which perhaps more clearly reveals the paradox, is 'This statement is false'. If the statement is true, then by what it reports it must be false. But if it is false, then what it reports is true.

The genius of Godel's Incompleteness Theorem is that it inserted a form of the Liar's Paradox into the *Principia*. By a technique referred to as Godel numbering, Godel was able to assign numbers to statement about numbers. Then, by the logical constructionism of the *Principia*, these statements could be equated to strings. If the strings could be produced, the statements were theorems. If the strings could not be produced, the statements were not theorems. The statement of interest to Godel was, 'this statement is not a theorem of the *Principia*'. If it is true, then the *Principia* cannot derive it. Regardless of the axioms used, such a paradox would always remain.

Mathematicians Understand—after 2,500 years of searching, they have been shown by one of their own that their perfect mathematical world is a sky castle—mathematics cannot provide the foundation that attaches itself to the real world. Although third order partial differential equations will still need to be solved to send the next Voyager to Pluto, there is a concrete limit to the claim that mathematics can place on truth. No finite system of axioms can ever be complete—Man's Ignorance Is Inherent

By providing an axiomatic basis for all that is currently known in mathematics (and by banishing self-containing sets from Set Theory, and self-referential statements from Logic) Russell and Whitehead salvaged 99.9999999% of the universality of their system. But Gödel's work has permanently changed our philosophical view of mathematics. It has also opened new branches of study in logic and mathematics. Mathematicians, for the most part, are once again happily back inside their Neo-Platonist closets playing with their numbers, points, equations, and so on. Still the curious breed, they want, as far as is possible, to know what they are talking about- and they are still willing to go to extraordinary lengths to be certain.

However, in the words of Bertrand Russell,

> "Pure Mathematics consists entirely of such asseverations as that, if such and such a proposition is true of anything, then such and such another proposition is also true of that thing. It is essential not to discuss whether the first proposition is really true, and not to mention what the anything is of which it is supposed to be true. If our hypothesis is about anything and not about some one or more particular things, then our deductions constitute mathematics. Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true."

Epilog: I believe the field of neuro-psychology can provide the axiomatic foundation to attach mathematics to the real-world (at least on the human scale).