ETHNOMATHEMATICS
A Multicultural View of Mathematical Ideas

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c h a p t e r t w o

Tracing Graphs in the Sand

In about 1905 a European ethnologist studying among the Bushoong in Africa was challenged to trace some figures in the sand, with the specification that each line be traced once and only once without lifting his finger from the ground. The ethnologist was unaware that a group of people in his own culture were keenly interested in such figure tracing. And, being unfamiliar with what Western mathematicians call graph theory, he was also unaware of how to meet the challenge.

The Bushoong who challenged the Western ethnologist were children concerned with three such figures. Among the Tshokwe, who live in the same region in Africa, sand tracing is not a children's game and hundreds of figures are involved. We will discuss both of these cases as well as a third from another part of the world, which is quite separate and more elaborate. Not only is each case from a different culture but their contexts within the cultures differ markedly. The mathematical idea of figures traced continuously is the central thread of this chapter. But for each culture there are, as well, other geometric and topological ideas involved in the creation, regularities, or relationships of these spatial forms. These too will be discussed, as our primary interest is mathematical ideas and, as with any ideas, they occur in complexes that do not necessarily conform to particular Western categories. To begin, however, a few ideas from graph theory need to be introduced.

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Graph theory, described geometrically, is concerned with arrays of points (called *vertices*) interconnected by lines (called *edges*). This field has been growing in importance in our culture because it provides new approaches, stimulates new concepts, and has many applications. Graphs are particularly useful in studying flows through networks. For example, traffic flow involves intersections (considered to be the vertices) interconnected by roads (considered to be the edges).

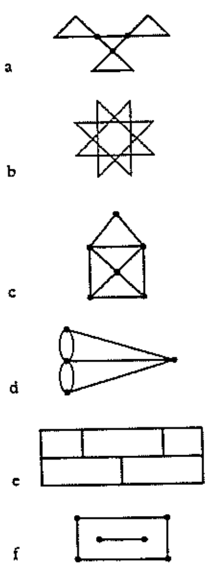

Figure 2.1. Graphs

A few graphs are shown in Figure 2.1. In graph theoretic terminology, Figures 2.1a–2.1e are said to be *connected planar* graphs. A *connected* graph is one in which each vertex is joined to every other one via some set of edges. (In contrast to the others, Figure 2.1f is not connected.) A *planar* graph is one that lies entirely in the plane; that is, it need not be depicted as rising out of this flat paper. A freeway overpass and the road beneath it, for example, would not be represented by a planar figure.

A classical question in graph theory is: for a connected planar graph, can a continuous path be found that covers each edge once and only once? And, if there is such a path, can it end at the same vertex as it started? This is the question that is said to have inspired the beginnings of graph theory by the mathematician Leonhard Euler. According to the story, there were seven bridges in Königsberg (then in East Prussia), where Euler lived. The bridges spanned a forked river that separated the town into four land masses. The townspeople were interested in knowing if, on their Sunday walks, they could start from home, cross each bridge once and only once and end at home. Euler showed that for the particular situation such a route was impossible and also started considering the more general question. Between Euler in 1736 and Hierholzer some 130 years later, a complete answer was found. To state the result another term

is needed, namely the *degree* of a vertex. The degree of a vertex is the number of edges emanating from it; a vertex is odd if its degree is odd and even if its degree is even. The answer to the question, first of all, is that not all connected planar graphs can be traced continuously covering each edge once and only once. If such a path can be found, it is called, in honor of Euler, an *Eulerian path*. Such a path exists if the graph has only one pair of odd vertices, provided that the path begins at one odd vertex and ends at the other. Also, such a path can be found if all of the vertices are even and, in this circumstance, the path can start from any vertex and end where it began. The graphs for which there cannot be Eulerian paths are those that have more than one pair of odd vertices.

With these results in mind, look again at the graphs in Figure 2.1. Graph a and graph b have all vertices of degree 4. Each can, therefore, be traced continuously covering every edge once and only once, beginning at any vertex and ending at the same place. Graph c has six vertices, two of degree 3 and four of degrees 2 or 4. It, therefore, has an Eulerian path that begins at one odd vertex and ends at the other. For graph d and graph e, no Eulerian paths exist as the former has four odd vertices and the latter eight.

Attempts to trace these five graphs are also of interest from a historical perspective. Graph d is a representation of the Königsberg bridge problem, with the vertices standing for the land masses and the edges standing for the bridges. Graphs a, b, c, and e link the domain of professional Western mathematicians to Western folk culture. Graph c may well be familiar to you, as tracing it is a children's street puzzle in many cities, from New York to London to Berlin. Graphs a, b, and e are from a collection of nineteenth-century Danish party puzzles. What is more, graph b, entitled "the nightmare cross," was said to have magical significance. Of these graphs, graph e is perhaps the most ubiquitous; not only was it a folk puzzle in nineteenth-century Denmark, but it has also appeared and reappeared in mathematical treatises since at least 1844. And the eminent philosopher Ludwig Wittgenstein, in discussing the very foundations of mathematics, used the problem of tracing a quite similar figure as one that captures the essence of the subject.

With this brief background, we return to the figure tracing challenge posed by the Bushoong children.

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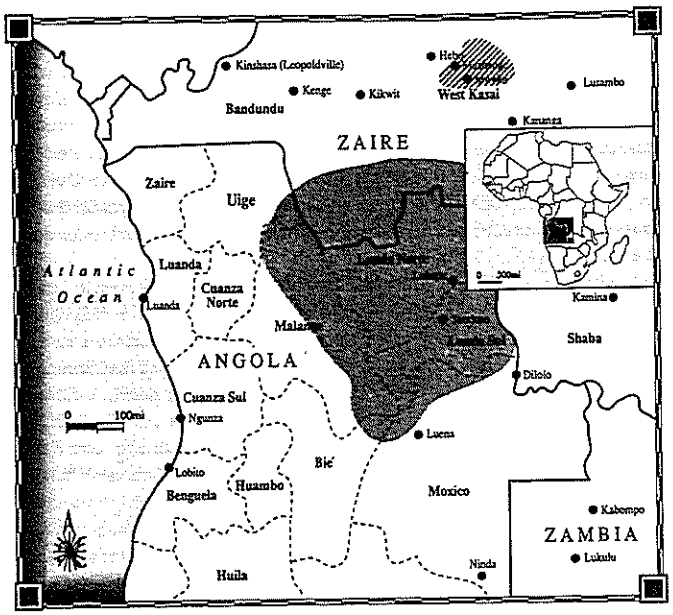
The Bushoong are one subgroup of the Kuba chiefdom. The chiefdom consists of at least four different ethnic groups separated into some fourteen subgroups. The hereditary Bushoong chief is the *nyimi*, the sovereign chief among all the chiefs, and so the Bushoong live in and around the Kuba capital (see Map 1). In the late nineteenth century, there were about 100,000 Kuba with about 4000 people living in the capital. Those living in the capital were the *nyimi*, hundreds of his wives, nobility, and specialized craftspeople. The Belgian government became the colonial authority just after 1910, ruling the Kuba territory indirectly until the establishment of Zaire in 1960.



Map 1. The Angola/Zaire/Zambia region. The crosshatched region is inhabited by the Bushoong, and the Tshokwe are in the majority in the shaded area.

In the Kuba system of exchange the Bushoong have the role of decorators; in particular, they are sculptors of wood and embroiderers of cloth. Every third day a market is held in the capital and the Bushoong obtain pottery, salt, meat, fish, ivory, brass, wood, and plain cloth from the other subgroups while supplying embroidered cloth, sculpted wooden objects, masks, woven belts, hats, and raffia velour. Decoration of daily utilitarian objects as well as ceremonial objects is done for more than its intrinsic aesthetic value. Self-decoration and the possession of

decorated objects bring prestige, and with prestige come political appointments and authority. Political position focuses around the *nyimi* and his appointed councils and titleholders; the former are men and the latter, both men and women. The relationship between decoration and political power is underscored by one element of the enthronement ceremony of the *nyimi*. At this ceremony he proclaims a special design that will be his official sign.

It is within this context, in a Bushoong village within the Kuba capital, that children play games drawing figures in the sand (see Figure 2.2). As with all Bushoong geometric decorations, each figure has a particular name. Naming, however, raises the very important point of cultural differences in perception. The name of a figure depends on how it is categorized. The Bushoong view a design as composed of different elementary designs, and the name given to a figure is the name associated with its most significant constituent. Thus, designs that appear the same to a Westerner may have different names, and those that appear different may have the same name. Moreover, significance is also related to process, and so sculptors may differ from embroiderers in the names they assign. Figure 2.2a is named *imbola*, which relates it to figures of the same name found carved on wooden objects, embroidered on cloth, and used as female scarification. In both name and appearance it is the same as a design formed of cowrie shells on a royal sash still in use in the 1970s. Figures 2.2b and 2.2c, named *ikuri banga* and *jaki na nyimi*, have no immediate counterparts in other figures but do look somewhat like some embroidered designs.

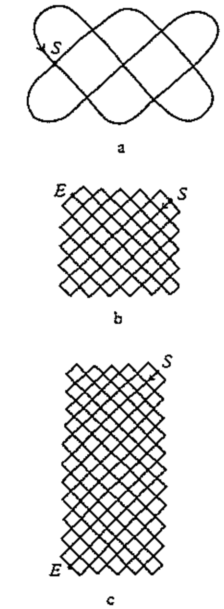


Figure 2.2. Bushoong sand figures. S and E denote the starting and ending points of the Bushoong tracings.

Figure 2.2a has seven vertices, each with four emanating edges. It can, therefore, be traced continuously starting anywhere and ending where one started. The starting and ending point used by the Bushoong children in their continuous tracing is marked by our label S on the figure. There is no record, however, of the specific tracing path that they used. When a continuous path is possible, there are many different ways that it can be accomplished. To emphasize the diversity, four possible paths are shown in Figure 2.3. Follow each of them with a pointer or your finger. You will see that Figure 2.3a first completes one of the triangular components and then the other, while Figure 2.3b goes around the exterior edges and then around the interior edges. Figures 2.3c and 2.3d, on the other hand, have no readily apparent plan. Other paths are also possible. How one "sees" the figure affects the path taken or, conversely, the path taken affects how one "sees" the figure.

Figures 2.2b and 2.2c each contain two vertices from which emanate an odd number of edges. From graph theory we know that tracing the figure continuously requires starting at one of the odd vertices and

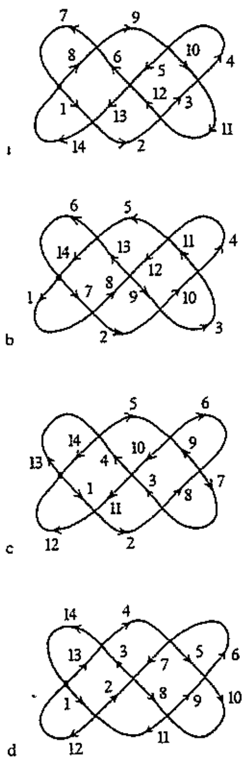


Figure 2.3. Some possible tracing paths

ending at the other. The Bushoong children, somehow, were aware of this as can be seen from their beginning and ending points (marked by our labels S and E on the figures). What is more, the tracing procedures they used for these two figures are known to us and we can see that they used a system and what that system was. For Figure 2.2b, starting at the uppermost right edge, proceed diagonally down to the left as far as possible, then up left as far as possible, up right as far as possible, then down right as far as possible, and so on going down left (dl), up left (ul), up right (ur), down right (dr). Because of the systematic nature of their drawing procedure, a concise set of instructions in terms of direction and number of units in that direction can be stated:

dl 10
ul 1, ur 10, dr 2, dl 9
ul 3, ur 8, dr 4, dl 7
ul 5, ur 6, dr 6, dl 5
ul 7, ur 4, dr 8, dl 3
ul 9, ur 2, dr 10, dl 1
ul 10.

By observation of the patterned number of units in each direction on successive sweeps, their procedure can be summarized as:

dl 10
$\begin{cases}\text{for } i = 1, \text{ then } 3, \text{ then } 5, \text{ then } 7, \text{ then } 9, \\ \text{ul } i, \text{ ur } 11 - i, \text{ dr } i + 1, \text{ dl } 10 - i\end{cases}$
ul 10.

If we were to trace a similar figure where there are, in general, N units rather than 10 units along the first diagonal, N would have to be an even number. Then the instructions, stated in terms of this general N, would be:

dl N
$\begin{cases}\text{for } i = 1, \text{ then } 3, \text{ then } 5, \text{ then 2 more each time until } N - 1, \\ \text{ul } i, \text{ ur } N + 1 - i, \text{ dr } i + 1, \text{ dl } N - i\end{cases}$
ul N.

The tracing procedure used by the Bushoong for Figure 2.2c is a bit more complex. Again the tracing starts at the uppermost right edge and

proceeds diagonally down to the left. As before, directions are changed after going as far up or down as possible. The procedure, in terms of number of units in each direction, is:

dl 10, dr 11
dl 1, ul 10, ur 11, ul 2, dl 9, dr 11
dl 3, ul 8, ur 11, ul 4, dl 7, dr 11
dl 5, ul 6, ur 11, ul 6, dl 5, dr 11
dl 7, ul 4, ur 11, ul 8, dl 3, dr 11
dl 9, ul 2, ur 11, ul 10, dl 1, dr 11
dl 10.

It is particularly noteworthy that both tracing procedures used by the Bushoong are systematic and the systems are variations on a like theme.

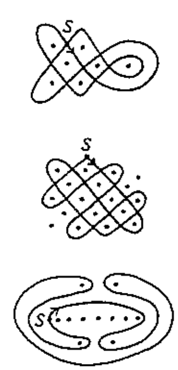
4

For the Tshokwe, the drawing of continuous figures in the sand is part of a widespread storytelling tradition. The figures, called *sona* (sing. *lusona*), are drawn exclusively by men. Because of the disintegration of native cultures brought about by colonialism, it is primarily older men who are knowledgeable and proficient in the drawing skill. The skill combines the memory of the drawings, the flowing movement of the fingers through the sand, and the "added art of storyteller who keeps his audience in suspense, intriguing them with the arabesques and holding them breathless until the end of the story." To draw the *sona*, an array of dots is first constructed. Great care is taken to create equal distances between the dots by simultaneously using the index and ring fingers and then moving one of these fingers into the imprint left by the other. The continuous figure is drawn surrounding the dots without touching them. (In some very few cases, disconnected additional lines are drawn through some of the dots.) For several figures, the beginning and ending points of the Tshokwe tracings have been recorded but the exact tracing paths are known for very few. Before we examine some of the *sona*, we present aspects of the Tshokwe culture that help to comprehend the figures and their associated stories and names.

The Tshokwe are in the West Central Bantu culture area. Historically, culturally, and geographically they are closely related to the Lunda, Lewena, Mbangala, Minungu, and Luchazi. Since the late nineteenth century, the Tshokwe have been the dominant majority in the Angoleis Lunda Sul/Lunda Norte regions (and just above that into Zaire) and spread more thinly in nearby areas (see Map 1). In 1960 this cluster of cultures contained about 800,000 people, of whom about 600,000 were Tshokwe.

The Tshokwe live in small villages under a family chief. The chiefdom passes to his younger brothers and then to his sisters' sons. His symbol of authority, which is also passed on, is a bracelet. The spirits of the ancestors (*mahamba*) and of nature are intermediaries between people and the Supreme Creator. The *mahamba* of the family of the community are represented by trees erected behind the chief's house and guarded by him. They are regularly given sacrifices and gifts of food. The individual families have smaller symbolic trees behind their own dwellings and small figurines are in cases or miniature huts around the village.

The *akishi* (sing. *mukishi*) are spirits that are incarnated in masks designated by the same name. When wearing a mask, the wearer is dedicated to and merged with the spirit. Many masks are associated with *mukanda*, a rite of passage of boys into adulthood. The rite begins when the chief of a village and his counselors decide there is a sufficiently large group of children. A camp, also called *mukanda*, is constructed in a specially cleared place. It is enclosed by a fence and contains round straw huts. After the ritual circumcision there are ritual dances, foods, clothes etc. The initiates then live in the camp for a year or two or even three, where they are taught, for example, rituals, history, and maskmaking. The mask Kalelwa, who had given the signal for the children to leave their homes, is in charge of the coming and going from the camp. The mothers are not allowed to see their sons and so Kalelwa makes loud noises to warn them away. The fathers can bring food and drink to the camp. After the prescribed education is complete, another celebration is held at which the young men are given new names, adult clothes, and then return home.



igure 2.4. *Sona* related the *mukanda*: a, the object circumcision; b, subject ıclear; c, the camp.

Several of the *sona* refer to the *mukanda*; three of these are shown in Figure 2.4. The simplest figure of all the *sona*, a continuous closed curve with no intersections, includes a story and exemplifies a Tshokwe topological concern. The figure (Figure 2.4c) depicts the camp. The line of dots are the children involved in the ceremony, the two higher dots the guardians of the camp, and the lower dots various neighbors or people not involved in the ceremony. The children and guardians are *inside* the camp; the others *outside*. One version emphasizes that the children cannot leave the camp, the other that the uninvolved people

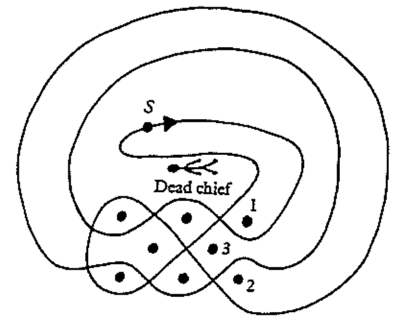


Figure 2.5. The succession of the dead chief. Dots 1, 2, and 3 are thieves. Only dot 3 can reach the village and steal the insignia.

cannot enter. A still lengthier version describes the fathers trying to bring food to the children while the *akishi* make ferocious noises in an attempt to frighten them away. No matter which version, most crucial to the story is that the figure, a simple closed planar curve, determines two regions of which it is the common boundary. That is what Western mathematicians call the *Jordan curve theorem.*

In other *sona* as well, the figure defines several regions in the plane and the story causes the audience to become aware that certain dots are interior or exterior to particular regions. The stories animate the dots in such a way that they become representative of any point within their region. For example, in the *lusona* shown in Figure 2.5, the chief of a village dies. Three thieves (dots labeled 1, 2, 3) try to steal the succession. The first two find that they cannot reach the chief but the third gets to the village, steals his insignias, and so succeeds him. Another *lusona* concerned with the separation of regions is actually a pair of *sona*. They attract our special attention because, in graph theoretic terms, they are *isomorphic*. A graph is defined by its vertices and interconnecting edges. It is the fact that two vertices are connected by an edge that is significant, not the length or curvature or color of the edge. Therefore, whether visually different or visually the same, two configurations with the same set of vertices similarly interconnected are graphically the same. To be more precise, two graphs are called *isomorphic* when their vertices and edges can be placed in one-to-one correspondence. Figure 2.6 shows a pair of graphs that differ visually but are isomorphic. The corresponding vertices have been labeled A, B, C, D, E and the same set of connecting

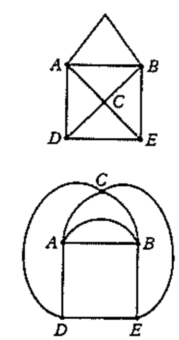
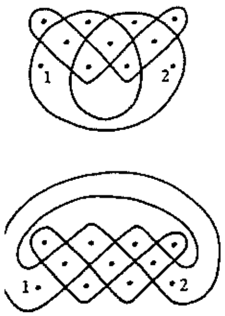


Figure 2.6. A pair of isomorphic graphs



igure 2.7. A pair of omorphic *sona*. Dots 1 and represent Sa Chituku and s wife, Na Chituku; the her dots are their ighbors. Sa Chitiku builds urriers that isolate his wife om the neighbors so that ie will attend to cooking stead of visiting.

edges *AB*, *AB*, *AC*, *AD*, *BC*, *BE*, *CD*, *CE*, *DE* are found in both. The fact that the *sona* in Figure 2.7 are isomorphic is especially noteworthy because they share the same story and the same name. Thus, our mathematical notion of isomorphism is reflected in a Tshokwe conceptual linkage. A pervasive characteristic of the *sona* is the repetition of basic units variously combined into larger figures. The *muyombo* trees representing the village ancestors are the *sona* shown in Figure 2.8. The uppermost row of dots represents members of a family praying to the ancestors before undertaking some action; the rest of the dots are all the houses of the village showing that the entire life of the clan is enveloped by the influence of the ancestral spirits. The same *lusona* is found in

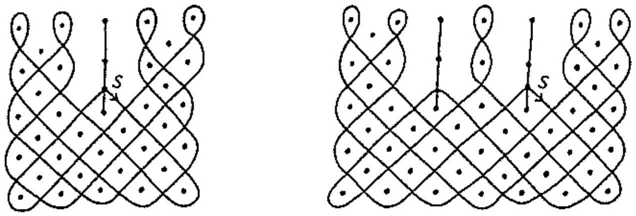


Figure 2.8. *Muyombo* trees

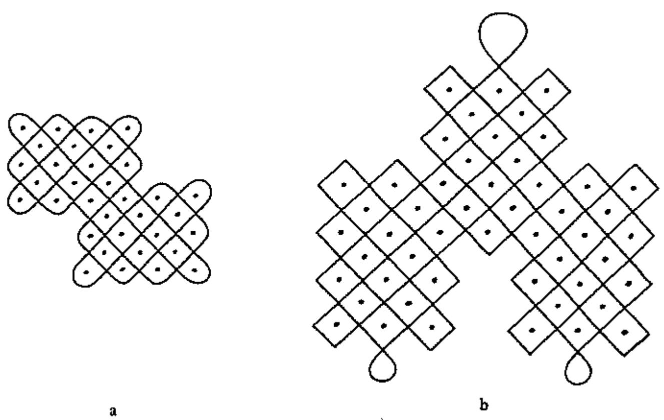


Figure 2.9. *Sona*: a, a hyrax in a rock; b, a leopard with his cubs

one-, two-, and three-tree versions. Comparing them, it is as if the smaller has been slit along its line of symmetry and another symmetric strip inserted. A different mode of extension is seen in a set of figures that has essentially one basic unit rotated and overlapped two, three, four, five, or six times. (Figure 2.9 shows those with two and three repetitions.) The *sona* in Figure 2.10 is unusually prominent and is even found in figures drawn on Tshokwe house walls. A set of *sona* in which four, six, eight, or nine of these units are variously connected are in Figure 2.11. The prominence of this *sona* is especially interesting because, without its dots, it is the very figure called *imbola* by the Bushoong children.

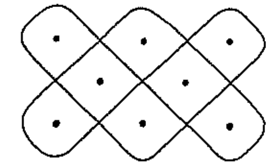


Figure 2.10. A small animal that lives in a tree hole and pierces the intestines

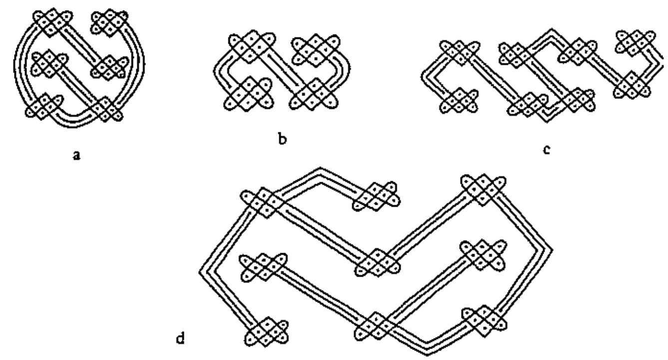
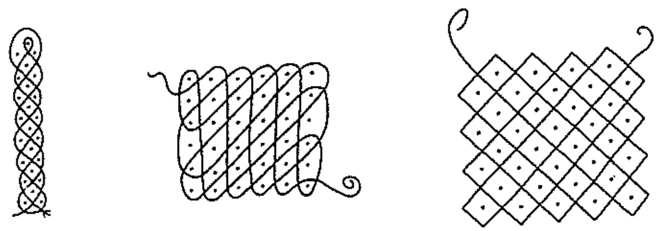


Figure 2.11. *Sona*: a, the *muyombo* tree of a former chief; b, the sanctuary of the *mahamba*; c, salt marshes; d, a labyrinth used in witchcraft

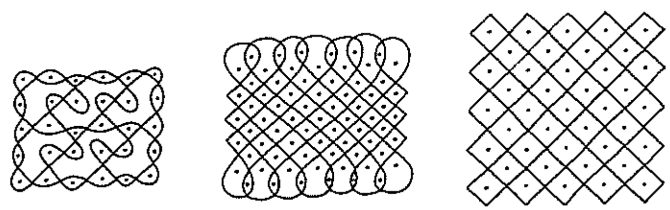
With few exceptions the Tshokwe *sona* are *regular* graphs of degree 4. (When all the vertices of a graph are of the same degree, the graph is *regular* and of that common degree.) Among the exceptions are some for which there are a pair of odd vertices. And, the fact that one of these corresponds to another of the three drawn by the Bushoong children, with the challenge that it be continuous with no retracing, provides further indication that there is some connection between the graphic concerns of the two groups. Three of the *sona* with a single pair of odd vertices are shown in Figure 2.12.

Ignoring the two flourishes, Figure 2.12c is the figure for which a general drawing procedure was stated earlier. The number of units along the first diagonal was designated by *N*; for the Bushoong figure *N* was equal to 10 while here N equals 8. Another Tshokwe figure is quite similar; the primary difference is that now N equals 6. Thus these constitute a set of three figures that share an overall configuration but differ due to the variation of a single parameter. Examples from three other sets are shown in Figure 2.13. Figure 2.13a is drawn on a 5×6 grid of dots; the other member of its set has the same configuration but is drawn on a 9×10 grid. The grid sizes for the set can be described as $(N - 1) \times N$, where N is 6 for the former and 10 for the latter. The *lusona* in Figure 2.13b is from another set of configurations drawn on grids of N rows of N dots each alternating with $N - 1$ rows of $N + 1$ dots each. Here N is 6, and for the other member of the set N is 7. Figure 2.13c comes from an even larger set of *sona*: these figures have rows of N dots alternating with rows of $N - 1$ dots for a total of $2N - 1$ rows. For Figure 2.13c, $N = 5$ and for other members of the set, $N = 2$, 6, and 8.



a b c

Figure 2.12. *Sona*: a, a large snake; b and c, intertwined rushes



a b c

Figure 2.13. *Sona*: a, the marks on the ground left by a chicken when it is chased; b, fire; c, captured slaves surrounded during a night encampment

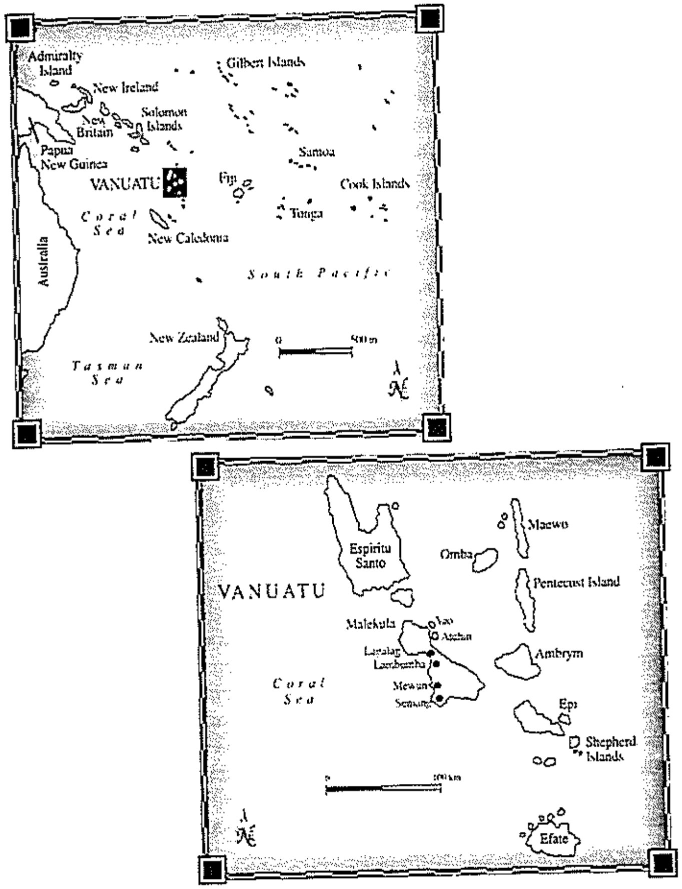
Some summarizing generalizations can be made based on the Tshokwe *sona* as a group.

1. Clearly the Tshokwe are interested in drawing figures continuously and in the use of curves to separate regions of the plane.
2. Dot placement, significant because it foreshadows the final figure, is an important part of the drawing procedure. Careful attention is given to the creation of rows and columns of equally spaced dots.
3. There are sets of *sona* that are similar in structure but varied in size. One type of set is characterized by a subfigure that is systematically repeated a different number of times in each *lusona* in the set; the other type is characterized by having a variable dimensional parameter.
4. The overriding feature of all of the Tshokwe *sona* is the prevalence of regular graphs of degree 4. Those few that have a pair of vertices of odd degree share another aspect: all other vertices in the figure are of degree 4. And what's more, the pair of odd vertices is *nonessential*; that is, a line can be drawn connecting the odd vertices without involving any other vertices or intersecting any edges of the figure. In short, they, too, are very close to being regular graphs of degree 4.

5

We now turn to another culture in another part of the world. Here, too, there is a tradition of tracing figures in the sand. The people are different; the figures are different; and the relationship of the figures to the culture is different. The concern for Eulerian paths is still central but the associated geometric and topological ideas are markedly different. What makes this case even more distinctive is that the exact tracing paths for the majority of figures are known to us. The ethnologist who first noted this figure tracing tradition, believing there was something most unusual in it, was meticulous in recording the specifics of almost 100 figures. This crucial information enables us to go beyond speculation or generalities and to see beneath the external features of completed figures.

The Republic of Vanuatu, called the New Hebrides before its independence in 1980, includes a chain of some eighty islands stretching over about 800 kilometers, with an indigenous population of close to 95,000. Malekula, one of the two largest islands, constitutes about one-fourth of Vanuatu's area. Europeans began to exploit this island about 1840. The social and psychological effects of colonial rule and the introduction of European diseases reduced the population by about 55 percent between the 1890s and 1930s. The culture area of interest here



Map 2. Vanuatu in the South Pacific

includes principally Malekula and the islands of Vao and Atchin, just off its coast, but also extends to the nearby islands of Omba, Pentecost, and Ambrym (see Map 2). While there are some differences among them, all will be included under the Malekula.

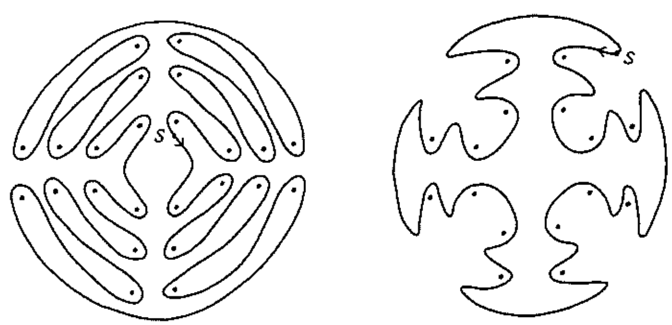
The drawing of continuous figures is tightly enmeshed in the Malekulan ethos. Many of the figures are named for local flora and fauna but several are related to important myths and rituals. In addition to illustrating myths, there are even myths in which the drawing concept plays a significant role. The drawings, called *nitus,* are executed in the sand by men and knowledge of them is handed down from generation to generation. Often a framework of a few horizontal and vertical lines or rows of dots precedes the drawing but is not considered a part of the figure and, occasionally, lines for tails or such are added at the end. The figure is to be drawn with a single continuous line, the finger never stopping or being lifted from the ground, and no part covered twice. If possible, the drawing is to end at the point from which it began. When it does, a special word is given to it: the figure is *suon.*

The pervasive aspects of life in Malekula that appear in descriptions of the *nitus* are graded societies and pigs. The graded societies define a person's rank in the overall society. Men's societies have from ten to thirty ranks within them; women's societies are less elaborate with only about three to five ranks. Each rank confers certain privileges including designs and ornaments that one is allowed to wear and places one is allowed to sit. Each advance is achieved through the construction of monuments and wooden drums, various payments, sacrifices of tusked pigs, and the performance of ritual acts. Advancement is accompanied by public dancing and feasting ceremonies. Passage from rank to rank becomes progressively more difficult and indicates an increase in power, influence, fertility, and the aid of supernatural beings. The acquisition and breeding of tusked pigs is of great importance because they are needed in most ceremonies and rituals and they serve as payment for special services by sorcerers or craftsmen.

Among the Malekula, passage to the Land of the Dead is dependent on figures traced in the sand. Its exact place, its entrance, and who guards the entrance vary with the locale but generally the entrance is guarded by a ghost or spider-related ogre who is seated on a rock and challenges those trying to enter. There is a figure in the sand in front of the guardian and, as the ghost of the newly dead person approaches, the guardian erases half the figure. The challenge is to complete the figure which should have been learned during life, and failure results in being eaten.

Another myth describes the origin of death among humans. The myth centers around two brothers, Barkulkul and Marelul, who came to earth from the sky world. (They are two of five brothers collectively referred to as Ambat.) One day Barkulkul went on a trip but before doing so he took the precaution of enclosing his wife so that he would know of any intruder. In one version he uses a vine to make a spiderweb-like design on the closed door of the house; in another he loops a string figure around his wife's thighs. Marelul visits her during Barkulkul's absence and then improperly replaces the vine or string. Upon his return Barkulkul goes to the men's house and challenges all the men to draw figures in the ashes on the floor. Marelul's figure gives him away and Barkulkul kills him. Eventually Marelul returns to life but smells so badly from decay that others avoid him. He returns to the sky world where there is no death. Those that remain on earth are the ancestors of men who work for sustenance and eventually die. These tales provide the indigenous association of sand or ash tracings, string figures, vines, and spiderwebs. Some aspects of these tales are the subject of several figures. More significant, however, is that the tales emphasize the need to know one's figures *properly* and demonstrate their cultural importance by involving them in the most fundamental of questions—mortality and beyond death.

Viewed solely as graphs defined by vertices and edges, the *nitus* vary in complexity from simple closed curves to having more than one hundred vertices, some with degrees of 10 or 12. Of some ninety figures that

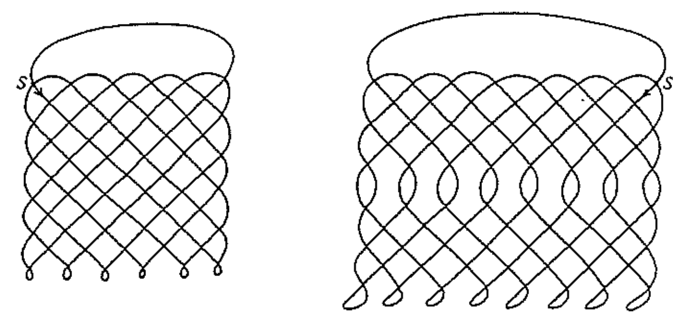


a b

Figure 2.14. a, a *nitus* associated with a secret society; b, a ghost

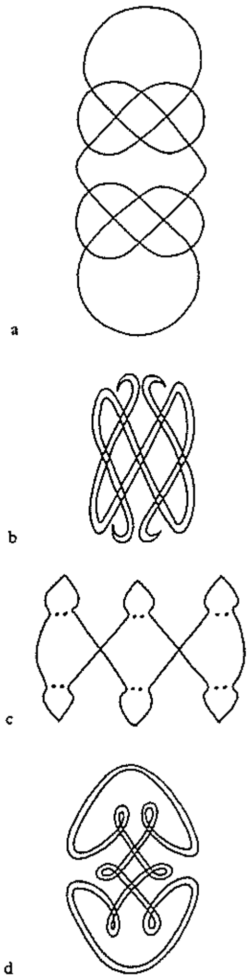
I have analyzed, about ten are simple closed curves and another twenty are regular graphs of degree 4. About thirty more, while not regular graphs, have only even vertices, and then about fifteen each have a single pair of odd vertices. In addition, because tracing courses were documented, there is ample corroboration of the Malekula's stated concern for what we call Eulerian paths. For those figures with known tracing courses, where Eulerian paths are possible, almost all were traced that way. And, similarly, the stated intention of beginning and ending at the same point was almost always carried out where that could be done. A story associated with a *nitus*, in which retracing of some edges does occur, highlights the fact that the Malekula consider backtracking to be improper. The *nitus* is called "Rat eats breadfruit half remains." First a figure described as a breadfruit is drawn completely and *properly*. Then the retracing of some edges is described as a rat eating through the breadfruit. Using the retraced lines as a boundary, everything below it is erased as having been consumed. Thus, the retracing of lines carries forward a story about an already completed figure but has the effect of destroying parts of it.

Figures 2.14–2.16 show a few of the *nitus* that are simple closed curves or regular graphs of degree 4. The lattice-like figures have tracing procedures that are similar to those of the Bushoong; that is, they systematically proceed diagonally down and up as far as possible. Here, however, the changes in direction involve loops and curved segments and



a b

Figure 2.16. *Nitus*: a, untranslated; b, the nest of the hawk with eggs in the middle and tail feathers projecting from the bottom



a b c d

Figure 2.15. *Nitus*: a, the path that must be completed to get to the Land of the Dead; b, a flower headdress, first made by Ambat, worn in a funeral ritual; c, three sleeping ghosts; d, related to two secret societies and women's initiation rites

there is a final sweep back to the starting point in order to end where one began.

Within individual figures the Malekula tracing procedures are systematic. But, far more important, there are larger, more general systems that underlie and unite groups of individual tracings. We will look at three of these extended systems, discussing each in terms of both the generalization that characterizes the system and some expressions of the generalization in particular *nitus*. The first two systems, taken together, account for the majority of *nitus* with a pair of odd vertices; the third involves figures that are more intricate in visual effect and in underlying concept. Keep in mind that tracing procedures are hand motions that leave transient marks in the sand. One way to talk about them or write about them is to refer to them by symbols. The symbols are ours, not theirs. But, as often happens in mathematics, if we assign our symbols with care, we can highlight logical and structural relationships that may otherwise escape us. It is the Malekula who created the tracing systems but it is we who are introducing symbols in an attempt to capture and convey the structure of their systems.

One set of six *nitus* is characterized by three basic drawing motions that are variously combined to give rise to a variety of figures. The *nitus* come from at least four islands and have diverse descriptions: a rat's track; a pig's head; a type of nut; the mark of the grandmother; octopus, food of the ghosts; and the stone on which is seated the ghost guarding the entrance to the Land of the Dead. Idealized versions of the three basic motions, denoted by the letters S, A, and B, are shown in Figure 2.17. Building with these basic notions, one circle-like unit results from S alone, a ladder of two circle-like units results from AB, a three-unit ladder from ASB, and a four-unit ladder from $AABB$. In general, an n-unit ladder results from $n/2$ A's followed by $n/2$ B's for n even; for n odd, an S is inserted between $(n-1)/2$ A's and $(n-1)/2$ B's. These ladders can be linked together in different ways. One *nitus* (Figure 2.18a) is the systematic connection of consecutively larger ladders (S, AB, ASB, $AABB$, $AASBB$) followed by consecutively smaller ladders ($AABB$, ASB, AB, S) resulting in a square array of circle-like units; another *nitus* (Figure 2.18b) is a rectangular array resulting from the systematic linkage of S, AB, AB, AB, AB, AB, AB, S. (In Figures 2.17 and 2.18, in order to identify the ladders on the *nitus*, each of the circle-like units is labeled i_j, meaning the

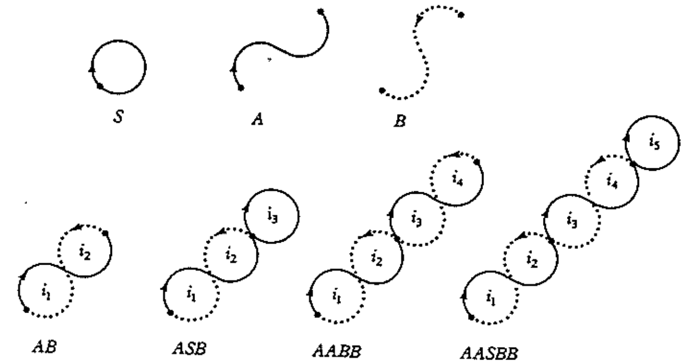


Figure 2.17. Basic motions S, A, and B combined into ladders of circle-like units. The subscripts denote the order in which the units are initiated.

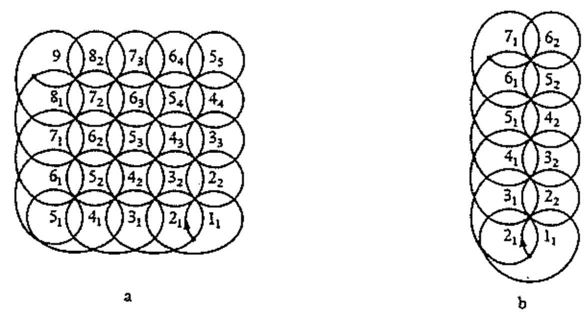


Figure 2.18. *Nitus*: a, the mark of the grandmother; b, a type of nut. Unit i_j is the jth unit initiated in the ith ladder.

ith ladder drawn in a specific *nitus* and the jth unit initiated in that ladder.) The other four *nitus* in the set are essentially the same except that they are built up from rotated versions of the three basic motions. And all of them differ slightly on their first and/or last few steps. Thus, in this system a few basic motions are formed into larger procedures, which are diversely yet systematically incorporated into still larger procedures.

The second tracing system underlies a second set of six *nitus*. The figures are from three islands and range in description from the nest of the burrowing ramé bird to a plaited coconut mat to a variety of yam. Each tracing begins with some initial procedure, that is, a particular sequence of drawing motions. Then, whatever the procedure, it is repeated within or around itself getting smaller and smaller or bigger and bigger. Figure 2.19 shows one of these *nitus* with its initial procedure. Another, less visually intricate because the repeated procedure never

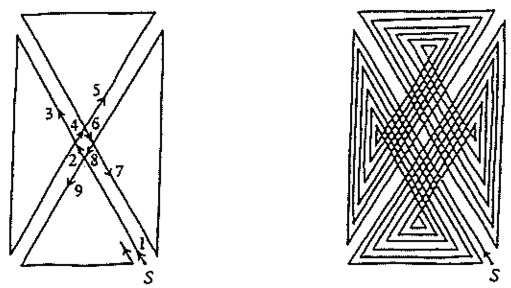


Figure 2.19. Nest of the *ramé* bird

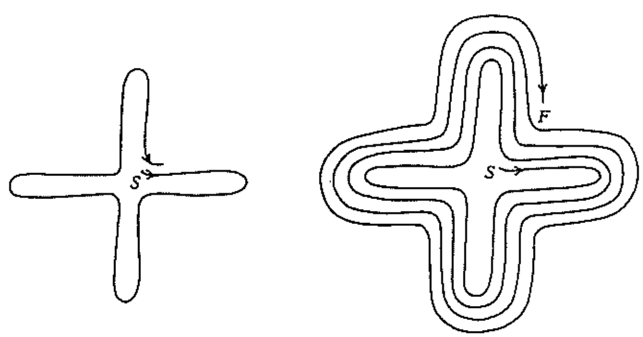


Figure 2.20. A type of yam

crosses a previously traced line, is shown in Figure 2.20. In the previous system, the same basic drawing procedures were combined in different ways; in this system it is the basic drawing procedures that differ while their manner of combination remains the same. The *nitus* in this group show the general tracing concept of an iterated procedure combined with systematic size modification. Also, the initial procedures, although different, share the feature of *almost* having bilateral or fourfold symmetry. In each, what interferes with the symmetry is the size modification just at the end of the procedure.

The third system is characterized by what I call a *process algebra*. The word *algebra* is used in its most fundamental sense: there are variable entities that are operated upon in accordance with specific rules. Here the variable entities are tracing procedures and the rules include processes that transform the procedures into other procedures. Furthermore, for the figures, only a distinct set of transformation processes is used. To give substance to this algebra, it is necessary to elaborate the specific set of processes involved. But first, since I am using them in a very specific way, the meaning of the words *procedures* and *processes* need to be made more explicit.

A tracing *procedure* is a particular sequence of motions used to draw a curve segment. It includes, therefore, both the directions of the motions and the order in which they occur. Figure 2.21a shows an example of a hypothetical tracing procedure; call it A. Procedure A is the sequence of motions: one unit up, one unit right, one unit diagonally up to the right.

A *process* is the way in which a tracing procedure is modified. If, beginning from the end of our hypothetical procedure A, the same procedure is repeated, the process is *identity* (no change), the resulting procedure is still A, and the overall procedure is A followed by A, which we symbolize by AA (see Figure 2.21b). If, however, when drawing the second segment every motion in A is rotated clockwise through 90° (up becomes right, right becomes down, diagonally up to the right becomes diagonally down to the right), the process is *90° rotation*, the new procedure is identified symbolically as A_{90}, and the overall procedure is AA_{90}. Another type of modification is the reflection of each motion across the vertical; that is, rights and lefts interchange while ups and downs remain the same . The result of a *vertical reflection* of A, which we denote by A_V, is shown in Figure 2.21d. You may have already noticed that these rotations and reflections look somewhat different from what we usually visualize when these words are used. That is because the procedure rather than the curve segment is being transformed, and the new procedure

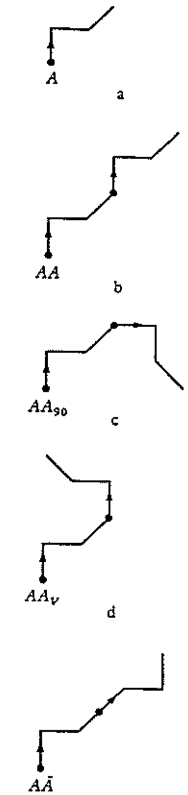


Figure 2.21. A few of the processes

begins where the previous one ended. Another of the processes modifies the order of the motions rather than their directions. This process, here called *inversion*, reverses the order of the drawing procedure; whatever motion was last becomes first and so on until first comes last. (We use $\bar{A}$ for the inversion of procedure A and $A\bar{A}$ is shown in Figure 2.21e.) In all, the processes of this algebra are: no change or identity (I); reflection over a vertical line (V); reflection over a horizontal line (H); rotation through 90°, 180°, or 270°; and each of these can be, but need not be, simultaneous with inversion ($^-$). There are, therefore, twelve processes, as illustrated in Figure 2.22.

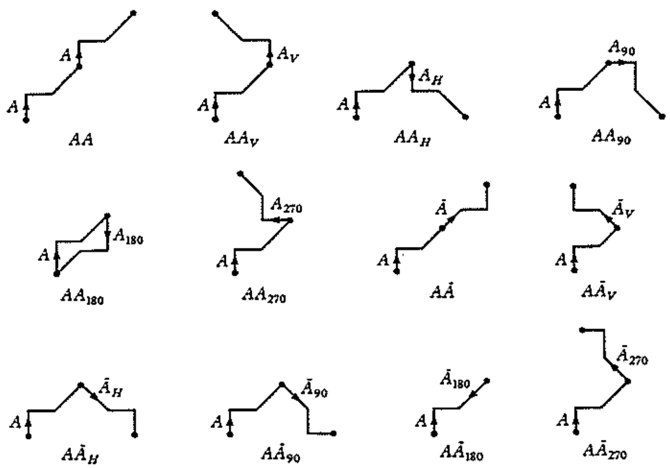


Figure 2.22. All of the processes

Now, turning to some individual figures in this group, each can be discussed in terms of the procedures and processes specific to it. For the *nitus* in Figure 2.23, when the initial procedure A (shown in the figure) is selected as the basic tracing unit, it becomes possible to concisely describe the overall tracing procedure as $AA_{90}A_{180}A_{270}$. Similarly, for the *nitus* in Figure 2.24, the overall procedure is $AA_{180}A_VA_H$ in terms of the initial procedure A we designated as its tracing unit.

Using the same basic tracing units, some alternative symbolic de-

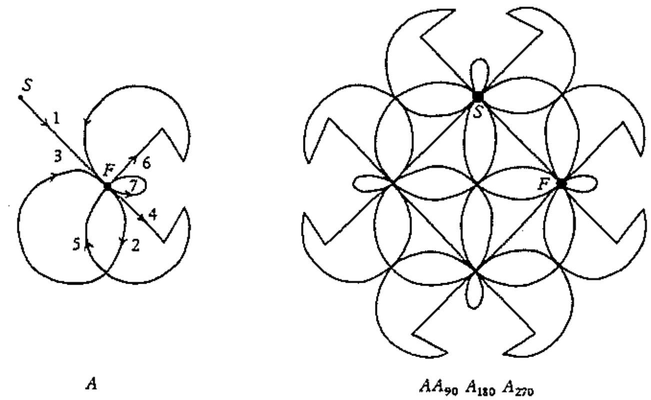


Figure 2.23. A yam

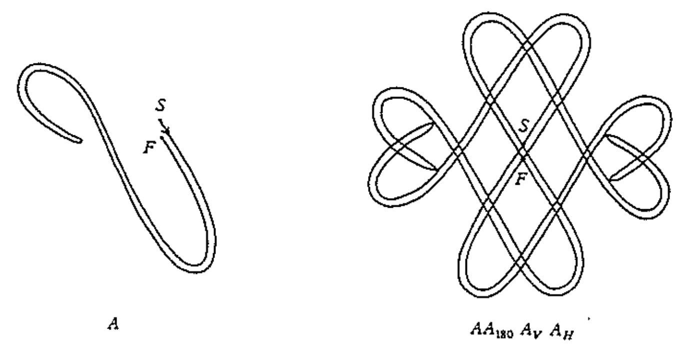


Figure 2.24. Untitled

scriptions are posssible. In each of the foregoing descriptions, successive procedures were phrased in terms of the initial procedures. Each could be phrased, instead, in terms of the procedure that is its immediate predecessor. For the first *nitus*, that is:

$$AA_{90}(A_{90})_{90}(A_{180})_{90} \quad \text{rather than } AA_{90}A_{180}A_{270}.$$

The new formulation rests upon the fact that a procedure modified by any process is itself a procedure that can, in turn, be modified. Thus the resulting procedure is the same whether one thinks of procedure A as having been rotated by 180° or thinks of procedure A_{90} as having been rotated by 90°. Another alternative is:

$$AA_{90}A_{180}A_{270} = (AA_{90})\underbrace{(AA_{90})_{180}}_{A_{180}A_{270}}.$$

This formulation reflects the following train of thought: procedure A is followed by procedure A_{90} and then their combined result is rotated by 180°. The formulation is possible because a procedure (in this case A) followed by another procedure (in this case A_{90}) is simply a new procedure (AA_{90}). These three symbolic formulations reflect three different conceptualizations but, nevertheless, all involve consecutive rotations. Taken together, the three versions highlight both the power and limitation of this algebraic approach. The power is that we can give expression to the structure of the overall tracing procedure. Further, symbolic manipulation enables different versions that reflect the same structure but different and equally plausible conceptualizations. The limitation, however, is that we have no way of selecting among different versions. In other words, the algebraic approach increases our understanding of what the Malekula actually did but does not uniquely translate into how they conceived of what they did.

Let us examine the description of Figure 2.24. For this, a little more symbolic manipulation is required. Again, the goal is to rephrase the description so that it is in terms of predecessor procedures rather than in terms of the initial procedure. That is, we will try to fill in the question marks for:

$$AA_{180}A_VA_H = AA_{180}\underbrace{(A_{180})_?}_{A_V}\underbrace{(A_V)_?}_{A_H} \quad \text{and in}$$
$$= (AA_{180})\underbrace{(AA_{180})_?}_{A_VA_H}.$$

We seek to find, for example, what process must act upon a 180° rotation to make the result appear the same as a vertical reflection. Table 2.1 is a

summary of the results of all possible process pairings, and so within it we can find the answer. The column headings are the first process and the row labels are the second process.

Table 2.1

	I	90	180	270	V	H
I	I	90	180	270	V	H
90	90	180	270	I	(Y)	(X)
180	180	270	I	90	H	V
270	270	I	90	180	(X)	(Y)
V	V	(X)	H	(Y)	I	180
H	H	(Y)	V	(X)	180	I

Entries in the table were generated by (and can be verified by) actually carrying out the paired processes and then comparing the result with the individual processes. Thus, for example, a 180° rotation acting upon a 90° rotation appears the same as a 270° rotation, while a 180° rotation acting upon a vertical reflection appears the same as a horizontal reflection. (The circled entries X and Y are not among the individual processes and so the pairings leading to them could not have occurred. Had X and Y been present, they would have corresponded to reflections across diagonal lines.) The inversion process is not included in this table as it acts independently; that is, any procedure first inverted and then transformed by some process gives the same result as would inversion of the already transformed procedure. (For more details about inversion, see the notes to this section.)

Now we can fill in the question marks related to Figure 2.24. To resolve $(A_{180})_? = A_V$, we seek a V in the column headed 180° and then identify the row. Thus, we find $(A_{180})_H = A_V$. Similarly, for $(A_V)_? = A_H$, since the column headed V and row labeled 180° have H in their intersection, $(A_V)_{180} = A_H$. To solve $(AA_{180})_? = A_VA_H$, we first clarify a formality that has been implied but should be explicitly stated, namely,

$$(AB)_P = A_PB_P.$$

This means that for any of the processes (P) in the table, the result is the same whether procedures A and B are carried out sequentially and the combined result transformed, or whether each is transformed individually and then they are carried out sequentially. For our purposes then,

$$(AA_{180})_? = A_V A_H \quad \text{can be rephrased as}$$
$$A_?(A_{180})_? = A_V A_H.$$

As we can guess and then verify in the table, this question mark should be replaced by a V. We have therefore found alternative equivalent descriptions focusing on initial procedure, preceding procedures, and preceding combination of procedures. They are:

$$AA_{180}A_V A_H = AA_{180}(A_{180})_H(A_V)_{180} = AA_{180}(AA_{180})_V.$$

Each provides a different conceptualization of the same overall procedure and, while the last seems more in keeping with some of the other tracings, choosing among the three would only be speculation.

Several elaborate figures are shown in Figures 2.25–2.28. Each is traced in three or four stages; that is, a continuous figure is drawn and then another, picking up from its endpoint, is drawn superimposed, and so on. Thus, each stage is a continuous figure in and of itself as is the total final figure. The interrelationship of the stages implies planning and a clear vision of the final goal. (In order to more fully savor the ideas of the Malekula, you are encouraged to try to trace some of these *nitus* without looking at the Malekula procedures. Then trace some of them using the Malekula procedures. Not only may you find their conceptions different from your own, but you may find many of their procedures quite graceful.) Figure 2.25 involves three stages, each a procedure and its 180° rotation. Its overall description is $AA_{180}BB_{180}CC_{180}$. (The first stage, viewed in finer detail, contains within it smaller procedures and their transforms: $A = X\bar{X}_V Y\bar{Y}_H$.) Another *nitus*, completed in four stages, is in Figure 2.26, described as $A\bar{A}B\bar{B}C\bar{C}D\bar{D}$. The processes of Figure 2.27 differ from stage to stage; this *nitus* has three stages and a small coda that is common to several of the figures. The stages are $(AA_V)(AA_V)_H$; $(BB_{270})(BB_{270}\text{truncated})_{180}$; and $C\bar{C}_V$. The emergence of such succint descriptions convinces us of the mathematical nature of the Malekula tracings. The procedures are both formal and logical but, what is more, they have structural simplicity while giving rise to complexity of visual effect.

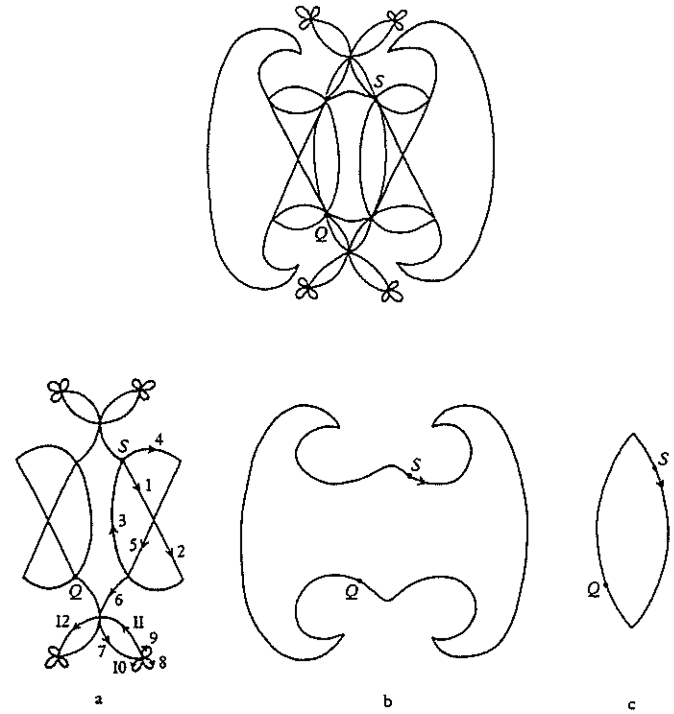


Figure 2.25. The *nitus* has three different names: "the stone of Ambat" and two others that refer to a story in which a mythical person attempts to kill and eat the Ambat brothers. Traced in three stages, each stage begins and ends at S. For each stage, the initial procedure goes from S to Q, and then the same procedure transformed by a 180° rotation goes from Q back to S. The complete procedure is $AA_{180}BB_{180}CC_{180}$, where $A = X\bar{X}_V Y\bar{Y}_H$ (X is segments 1–3 and Y is segments 6–11).

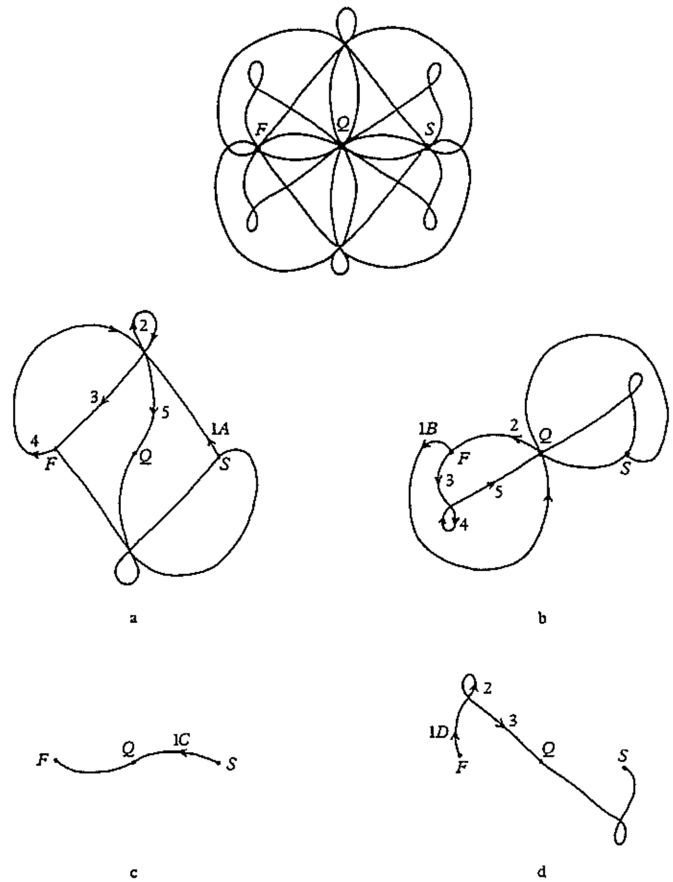


Figure 2.26. A turtle, traced in four stages. The first and third stages begin at S and end at F, and the second and fourth stages begin at F and end at S. For each stage, the initial procedure goes from S (or F) to Q, and then the same procedure transformed by inversion goes from Q to F (or S). The complete procedure is $A\bar{A}B\bar{B}C\bar{C}D\bar{D}$.

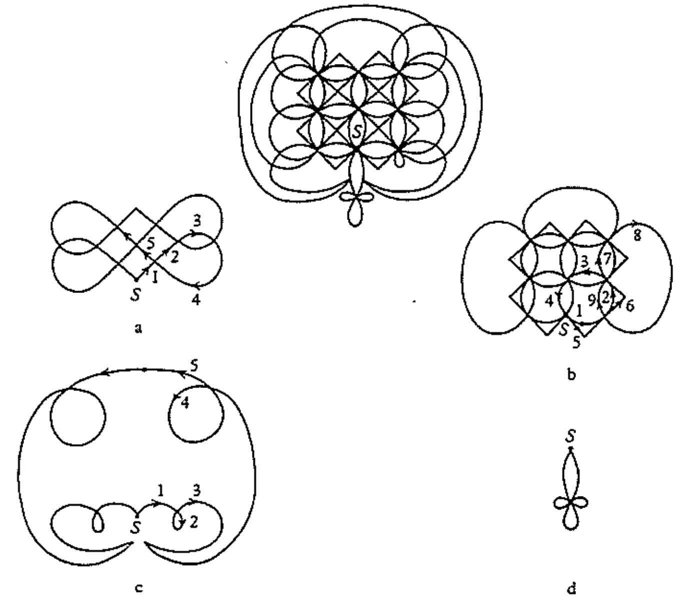


Figure 2.27. The banana stump, a tracing with three stages and a coda. Each stage begins and ends at S. The first stage is $(AA_V)(AA_V)_H$, where A is segments 1–5; the second stage is $(BB_{270})(BB_{270}$ truncated$)_{180}$, where B is segments 1–9; and the third stage is $C\bar{C}_V$, where C is segments 1–5.

This system encompasses a large number of the graphs with vertices of only even degree, including the very simple figures seen previously in Figure 2.14. The *nitus* in the group also have something else in common; with one exception, all are symmetric figures. Some have bilateral symmetry (a single axis of reflection) and most have double axis symmetry (a pair of perpendicular axes of reflection). Some of the latter also show 90° rotational symmetry and just one figure has 180° rotational symmetry without double axis symmetry. Here too, knowing the Malekula tracing procedures allows a deeper understanding. Rather than symmetry being an externally imposed concept based solely on *our* viewing of completed, static figures, symmetry can be seen as arising from

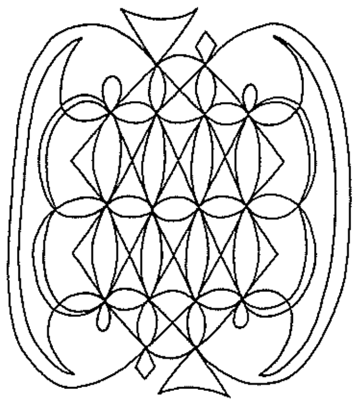


Figure 2.28. Two fishes head to tail. The complete procedure for this three-stage tracing is $AA_{180}BB_{180}CC_{180}$.

specific actions of the Malekula. This is especially significant because of the interrelationship between how one traces a figure and what one sees in it. Figure 2.23, for example, might be viewed as either double axis symmetry or fourfold rotational symmetry. In fact, it resulted from a tracing procedure involving rotation. Furthermore, we know (and are possibly surprised to learn) that the unit of rotation was the curve segment from initial procedure A. In Figure 2.25, the visually ambiguous case of double axis symmetry or 180° rotational symmetry was actually brought about by 180° rotation. Looking at the apparent case of double axis symmetry in the *nitus* of Figure 2.26, it is surprising to find that none of its stages share that effect. Each of them shows 180° rotational symmetry brought about by the process of inversion! In the only *nitus* that shows 180° rotational symmetry without double axis symmetry (Figure 2.28), the three-stage tracing procedure $AA_{180}BB_{180}CC_{180}$, the visual effect, and the figure's description (a pair of like fishes head to tail) all corroborate the 180° rotation.

7 Some summarizing generalizations can be made based on the corpus of *nitus*.

1. Of the *nitus* for which it is possible to use a single continuous line and the courses are known, almost all were actually traced that way. This coincides with the stated intention of the Malekula. Another stated intention is, if possible, to end where one began. Of the *nitus* where that is possible and the courses are known, almost all do just that. And, backtracking was intentionally avoided, as underscored by erasure in the case where it occurs.

2. Our graph theoretic distinction between even-degreed graphs and those with a pair of odd vertices is correlated with a substantial division within the *nitus*. They are divided by the systems used to trace them as well as by the presence or lack of visual symmetry.

3. Symmetry is obviously of considerable importance as a large majority of the *nitus* show some form of it. (Those that do not are mainly among the graphs with a pair of odd vertices.) Furthermore, since the observed symmetries result from tracing procedures that involve reflections and rotations, we can conclude that visual symmetries were intentional on the part of the Malekula.

4. Within the Malekula's self-imposed constraints of symmetry, continuous paths, and ending at the starting point if possible, the individual figures are traced systematically. Further, for different groups of figures, these systematic tracings are particular expressions of larger systems. Each of the larger systems combines and/or transforms basic procedures into other procedures in ways that are both general and formal. One systematic mode of combination is used for the lattice-like figures; another is the systematic repetition of three basic procedures to form ladders with different numbers of circle-like units, which are then systematically formed into rectangular arrays. A general tracing scheme is the use of stages for the more complicated *nitus*, where the stages are essentially subgraphs with constraints similar to the graphs themselves. Over and above all of these, and used alone or in conjunction with them, are the transformations described as the set of processes including rotation, reflection, and inversion and the similitude transformation by which the sizes of the shapes are diminished or enlarged. The elements of this overall system are procedures to create shapes made up of curved and straight line segments; the transformations modify the direction or order in which the procedure is carried out or its scale. The procedures and transformed procedures are variedly yet systematically combined into larger procedures, that is, into paths that trace the corpus of the *nitus*.

5. In addition, related to or emerging from these formal aspects, there is a clear aesthetic component that we can appreciate.

8 Whether in the context of games, story telling, traveling to the Land of the Dead, abstract line systems, or Sunday strolls in Königsberg, different peoples have pondered the same problem. Each culture surrounded the idea of figures traced continuously with other geometric and topological ideas. Widely separated in space and in traditions, each of the three cultures, nevertheless, found the idea sufficiently intriguing to elaborate it well beyond practical necessity. This chapter, building around the shared underlying mathematical concept, has directed attention to the differences in culture, differences in context, and, hence, differences in elaboration.

Notes

2. For a brief introduction to graph theory, see O. Ore, *Graphs and Their Uses*, New Mathematical Library 10, Random House, N.Y., 1963, and for its history, see N. L. Biggs, E. K. Lloyd, R. J. Wilson, *Graph Theory 1736–1936*, Clarendon Press, Oxford, 1976. The idealized model of the Königsberg bridge problem shown in Figure 2.1d is usually attributed to Euler. However, according to R. J. Wilson ("An Eulerian trail through Königsberg," *J. of Graph Theory*, 10 (1986) 265–275), it first appeared in 1894 in *Mathematical Recreations and Problems* by W. W. Rouse Ball. Figure 2.1e appeared in T. Clausen, "De linearum tertii ordinis proprietatibus," *Astronomische Nachrichten* 21 (1844) cols. 209–216 and in J. B. Listing, *Vorstudien zur Topologie*, Göttinger Studien 1, 1847, 811–875. The former noted that a minimum of four lines was required and the latter stated the more general case that a figure with $2n$ odd vertices requires n lines. It was not until 1891, again using this figure as an example, that a proof for the general case appeared in E. Lucas, *Récreations Mathématiques*, Vol. 1, Gauthier-Villars et Fils, Paris. Figures 2.1a, b, and e are from a collection of Danish folk puzzles detailed in J. Kamp, *Danske Folkeminder, Aeventyr, Folksagen, Gaader, Rim Og Folketro, Samlede Fra Folkemende*, Neilsen, Odense, 1877. Particularly note the interplay of professional literature, mathematical recreation books, and Western folk culture. The statement by L. Wittgenstein is on p. 174e of his *Remarks on the Foundations of Mathematics*, G. H. von Wright, R. Rhees, G. E. M. Anscombe, eds., Blackwell, Oxford, 1956.

3. The Bushoong figures were collected by the ethnologist Emil Torday. They are reported in E. Torday and T. A. Joyce, *Notes Ethnographiques sur les Peuples Communément Appelés Bakuba, Ainsi que sur la Peuplades Apparentées les Bushongo*, Annales du Musée du Congo Belge, Ethnographie, Anthropologie, Serie 3, vol. 2, pt. 1, Bruxelles, 1910 and in E. Torday, *On the Trail of the Bushongo*, Lippincott, Philadelphia, 1925. The drawing processes for Figures 2.2b and 2.2c are not explicitly stated but can be reconstructed from the asso-

ciated comments and figures in the former. My description of the Bushoong is drawn from Torday & Joyce (1910) and from M. J. Adams, "Where two dimensions meet: the Kuba of Zaire" in *Structure and Cognition in Art*, D. K. Washburn, ed., Cambridge University Press, N.Y., 1983, pp.40–55; Monni Adams, "Kuba embroidered cloth," *African Arts*, 12 (1978) 24–39; D. C. Rogers, *Royal Art of the Kuba*, University of Texas Press, Austin, 1979; J. Vansina, *Les Tribus Ba-Kuba et les Peuplades Apparentées*, Annales du Musée Royal du Congo Belge, Série in −8°, Tervuren, 1954; J. Vansina, *Le Royaume Kuba*, Annales du Musée Royal de L'Afrique Central, Série in −8°, Tervuren, 1964; and J. Vansina, "Kuba art and its cultural context," *African Forum*, 3–4 (1968) 13–17. The discussion in this section is similar to that in M. Ascher, "Graphs in cultures (II): a study in ethnomathematics," *Archive for the History of Exact Sciences*, 39 (1988) 75–95.

4. My description of the Tshokwe is drawn from M.-L. Bastin, *Art Decoratif Tshokwe I*, Publicacoes culturais no. 55, Museu do Dondo, Companhia de Diamantes de Angola, Lisbon, 1961; M.-L. Bastin, "Quelques oevres Tshokwe de musées et collections d'Allemagne et de Scandinavie," *Africa-Tervuren* 7 (1961) 101–105; A. Hauenstein, "La corbeille aux osselets divinatoires des Tchokwe (Angola)," *Anthropos* 56 (1961) 114–157; M. McCulloch, *The Southern Lunda and Related Peoples*, International African Institute, London, 1951. The *sona* have been collected and reported by a number of people. My analysis focused on those figures reported by more than one collector. My article, "Graphs in cultures (II)," cited above, contains detailed results of that analysis. See that article for additional details and for specific citations for the figures and associated stories used here. The *sona* in my study were from H. Baumann, *Lundabei Bauern und Jägern in Inner-Angola*, Würfel Verlag, Berlin, 1935; Th. H. Centner, *L'enfant Africain et ses Jeux dans le Cadre de la Vie Traditionelle au Katanga*, Collection mémoires CEPSI no. 17, Elizabethville, Katanga (Lubumbashi, Zaire), 1963; M. Fontinha, *Desenhos na Areia dos Quicos do Nordeste de Angola*, Estudos, ensaios e documentos no. 143, Instituto de Investigaçao Cientifica Tropical, Lisbon, 1983; E. Hamelberger, "Ecrit sur le sable," *Annales des Peres du Saint-Esprit*, 61 (1951) 123–127; G. Kubik, "Kulturelle und Sprachliche Feldforschungen in Nordwest-Zambia, 1971 und 1973," *Bulletin of the International Commission on Urgent Anthropological and Ethnological Research*, 17 (1975) 87–115; G. Kubik, "African graphic systems—a reassessment (Part II)," *Mitteilungen der Anthropologischen Gesellschaft in Wien (MAGW)*, 115 (1985) 77–101; G. Kubik, "*Tusona* ideographs—a lesson in 'objectivity' in interpretation," in *Archeolgical Objectivity in Interpretation 1*, World Archeological Congress 1986, Allen & Unwin, London, 1986, pp. 1–30; J. Redinha, *Peredes Pintades da Lunda*, Publicações culturais no. 18, Museu do Dondo, Companhia de Diamantes de Angola, Lisbon, 1953; and E. dos Santos, "Contribuiçao para o estudo das pictografias e ideogramas dos Quicos" in *Estudos Sobre a Etnologia do Ultramar Português* 2, Estudos, ensaios e documentos no. 84, Junta de Investigações do