What C·o·u·n·t·s

How Every Brain Is Hardwired for Math

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THE FREE PRESS

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Born to Count

A child, at birth, is a candidate for humanity; it cannot become human in isolation.

French psychologist Henri Pieron, 1959

I argued in Chapter 2 that everyone counts, not because everyone has the opportunity to learn this special skill, but because we are born with special circuits in our brains for categorizing the world in terms of numerosities. Whether a culture has formal arithmetical education or not, whether the lives of its members involve frequent use of numbers or not, indeed, whether its language contains special words for numbers or not, people will be able to carry out basic numerical operations.

This Mathematical Brain hypothesis is a claim about the relationship between nature and nurture in the domain of numerical abilities. I show it graphically in Figure 3.1. Very simply, nature provides an inner core of ability for categorizing small collections of objects in terms of their numerosities, which I have called the Number Module. For more advanced skills, we need nurture: acquiring the conceptual tools provided by the culture in which we live.

There are many precedents for specialized innate abilities. For example, many perceptual capacities are already working in the newborn baby: babies can see colours, distinguish speech sounds from non-speech sounds, and, within a few days or weeks, focus their eyes and see clearly. It is not surprising that newborns can use their five senses, but are they able to categorize the world in a more abstract way? We now know that newborns categorize the world in terms of

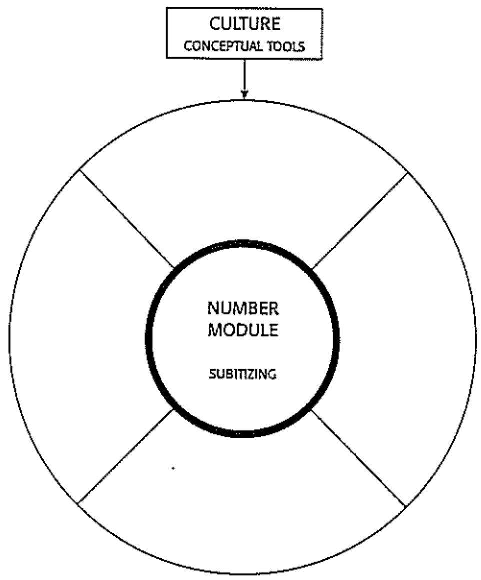

Figure 3.1 The newborn infant can use subitizing (the ability to tell numerosity at a glance, without counting) to discriminate and identify the number of things in a visual array, up to about 4. It has not, of course, had the opportunity to acquire conceptual tools such as counting words or finger patterns.

objects that have a continuous existence in time—if an object disappears, they stare in surprise. They also have expectations about how these objects will behave: one object cannot pass through another, for example. Babies will have had no opportunity to learn about these things: they seem to be built into the brain as part of the genetic code, and they kick into life as soon as the baby leaves the womb. However, being born with a capacity does not mean that it will be immediately evident. For example, we are all born with the capacity to grow pubic hair but it is not present in the newborn, appearing only after about 12 or 13 years of maturation.

The Piagetian Alternative

The influential Swiss psychologist Jean Piaget believed that our reasoning capacities emerged slowly, each new concept building on those previously developed. Numbers, he believed, are intimately connected with the development of the ability to reason logically and abstractly, though the ability for full abstract reasoning, in what Piaget called the stage of 'formal operations', was thought to emerge at about the same time as pubic hair.

> Our hypothesis is that the construction of number goes hand-in-hand with the development of logic, and that a pre-numerical period corresponds to a pre-logical level. Our results do, in fact, show that number is organised, stage after stage, in close connection with gradual elaboration of systems of inclusion (hierarchy of logical classes) and systems of asymmetrical relations (qualitative seriations), the sequence of numbers thus resulting from an operational synthesis of classification and seriation. In our view, logical and arithmetical operations therefore constitute a single system that is psychologically natural, the second resulting from a generalisation and fusion of the first, under two complementary headings of inclusion of classes and seriation of relations, quality being disregarded.[1]

He believed that our idea of numerosity was built on more basic capacities. These included the capacity to reason transitively—that is, the child should be able to reason that if *A* is bigger than *B*, and *B* is bigger than *C*, then *A* is bigger than *C*. Without this capacity, the child could not put the numbers in order of size, which is clearly fundamental. A second capacity the child must develop is the idea that the number of things in a collection is 'conserved', to use his technical term, unless a new object is added to the collection or an object subtracted from it. Merely moving the objects around—for example, spreading them out so they take up more room—does not affect number. Even more basic than either of these two capacities, as Piaget pointed out, is the ability to abstract away from the perceptual properties of the things in the collection. To grasp the numerosity of a collection, one needs to ignore all the particular features of the objects in it: their colour, their shape, their size, even what they are. A collection of three cats has the same numerosity as a collection of three chairs, or indeed of three wishes. The idea of number is abstract, very abstract. And the ideas of 'same number' or 'different numbers' are abstractions from abstractions.

According to Piaget, the emergence of the capacity for numerosity depends on the development of the necessary prior capacities, what Piagetians call 'prerequisites'. It also depends, as do many conceptual and logical abilities, on interacting with the world. The concept of numerosity could emerge as a result of manipulating objects, lining up collections to establish one-to-one correspondence between the members of the two collections, for sharing out sweets or toys.

The Empiricist Alternative

The philosopher Philip Kitcher makes the child's manipulation of objects in the environment the foundation of his approach to mathematics:

> I begin with an elementary phenomenon. A young child is shuffling blocks on the floor. A group of his blocks is segregated and inspected, and then merged with a previously scrutinized group of three blocks. The event displays a small part of the mathematical structure of reality, and may even serve for the apprehension of mathematical structure. Children come to learn the meanings of 'set', 'number', 'addition' and to accept basic truths of arithmetic by engaging in *activities* of collecting and segregating. Rather than interpreting these activities as an avenue to knowledge about abstract objects, we can think of the rudimentary arithmetical truths as true in virtue of the operations themselves.[2]

This approach is a version of 'empiricism', which claims that all ideas, all concepts, are derived from sensory experiences, and abstract ideas are constructed by generalizing from many particular experiences. On this account, children can have no grasp of numerical concepts until they are able to manipulate objects, segregate one group from others, scrutinize, remember the results of the scrutiny (which cannot be a number, since this is what they are setting out to explain), and merge these results with others (whatever this may mean). It is not clear at what age Kitcher believes that children are able to do these things, since he draws on no (empirical) studies of how children actually do acquire arithmetical concepts, but it cannot be earlier than about two or three years of age.

The approaches taken by both Piaget and Kitcher imply that categorizing the world is going to come quite late in the child's development. Piaget thought that the number concept emerged at about the age of four or five. After all, how can an infant—newly arrived in the world, unable to speak, and with the world just a buzzing confusion of colours and noises—use number to categorize what it sees or hears? How could it possibly add or subtract? The obvious, commonsense answer is that the infant cannot.

In this chapter, we shall see that the obvious answer is the wrong answer. The truth is much more surprising. Infants as young as the first week of life do indeed categorize the world in terms of numerosities, and infants of a few weeks, too young to have learned about arithmetic, can add and subtract. We shall see how counting provides the conceptual tools that take the child beyond what can be achieved by using the Number Module alone, beyond the ability to recognize numerosities, to the key idea of a sequence of numerosities ordered by size.

INFANT NUMEROSITIES

A baby just a day old is lying on her side in a hospital bassinet, quiet but alert. About 18 cm from her eyes is a white card with two circular black dots on it. She looks at the card. The card is replaced by another card with two similar black dots, but more widely spaced. The baby stares intently at it. The second card is replaced by the first, and the baby looks at it. All the while an experimenter, out of sight of the baby, is measuring how long she spends looking at each card. As the two cards continue to be presented alternately, the baby begins to lose interest and looks at the cards for shorter and shorter periods. Then a new card is presented, a card with three black dots in a line, and the baby immediately looks longer at this new card, twice as long in fact as at the previous card. Why? Does the baby just prefer looking at more black dots than at fewer? Two new cards are shown to the baby, one at a time. Each card now has three dots in a line, but on one of them the dots are more widely spaced. After a few trials the baby seems to lose interest and barely glances at the cards; then a new card is shown. This time with just two dots. The baby again looks longer at this new card. Is it just something new? To be new for the baby, she has to have a memory of what went before. What is represented in this memory? Is it the fact that there were black dots, or a white card? Is it the pattern of dots? Or could it be the *number* of black dots?[3]

This technique is known as 'habituation–dishabituation', and it makes use of the fact that babies like novelty and will look longer at new things: the same thing over and over again causes them to *habituate*—lose interest—while a new thing causes them to regain interest—*dishabituate*. What makes this technique particularly useful for studying the mind of the baby comes when you ask what counts as new for the baby. That is, what categories can the baby use for classifying her experiences? We know that newborns can make subtle distinctions among sounds. For example, they can categorize sounds into P sounds and B sounds. If they hear a series of P's they habituate, but when they then hear a B sound, they get interested again.[4] We know they can categorize by colour and by shape.[5]

But how do we know to what aspect of novelty the baby is responding in the dot task? It could be any new stimulus. Three American experimenters, Prentice Starkey, Rochel Gelman, and Elizabeth Spelke,[6] carried out a similar type of experiment with slightly older children, 6–8 months, but instead of black dots they used pictures of various objects: an orange, a key, a comb, an egg-beater, and so on. Instead of a card with two dots close together alternating with one with two dots far apart, each card had two objects, but different objects each time. The dishabituating card also had new pictures on it, but there were three pictures this time. Since each card was new for the babies, would they have mentally categorized the habituating cards as showing two things, so that when a card with three things was presented, they would regain interest and look longer? If they did look longer, it could not be because of mere novelty, since each card was new. It turned out that babies did look significantly longer at the card with three pictures on it.

Perhaps babies just like to see more pictures and this is why they looked longer. However, Starkey and his colleagues used the other sequence as well: a habituating series of cards with three pictures on them, followed by a dishabituating card with just two. Again the babies looked longer at the new number. The babies seemed to be sensitive to the number of pictures on the card. They categorized what they saw in a way that is quite abstract: the particular features of each picture—the colour, the objects depicted, their size, their brightness—which change with every card, have to be disregarded.

Stationary pictures of objects have a pattern: one object is a point, two objects will form a line, and three may form a triangle and four a quadrilateral. Perhaps babies are using some form of visual pattern perception rather than numerosity.[7] One fascinating study puts paid to that idea. Two Dutch psychologists, Erik van Loosbroek and Ad Smitsman from the Catholic University of Nijmegen, showed babies of 5 and 13 months not still pictures, but moving pictures. The pictures were of two, three, or four rectangles in shades of grey that moved in random trajectories on a computer monitor. From time to time one rectangle would appear to pass in front of another, occluding part of it. As in the previous studies, after a while the babies looked at the screen for less time, but when the number of rectangles was changed, either by adding one more rectangle or taking one away, they started to look for significantly longer. They cannot have been responding to a change in the pattern, since each of the rectangles was in constant motion, so they must have extracted the numerosity from the moving displays.[8]

We are beginning to find evidence that the infant's sensitivity to numerosity goes beyond collections of objects, still or moving. They are also sensitive to collections of actions. These are very different in kind from objects. Where does one action end and the other begin, and how are actions to be categorized? If I deal from a deck of cards, five cards to five players, is there one action—the deal—or twenty-five actions—each card?

The following remarkable experiment was carried out by Karen Wynn, a brilliant young researcher at the University of Arizona. The baby, this time about six months old, well before he has spoken his first words, is looking at a display stage with a puppet on it. In the habituation phase he sees the puppet make two jumps, with a short pause between the jumps. As with repeated presentation of two pictures, so the looking time gets shorter with each jump. The puppet makes three jumps, and the looking time nearly doubles. Again the reverse was used as a control: three jumps followed by two jumps.[9]

I have focused on numerosities of visible collections, but numerosities are supposed to be abstractions, distinct from perceptual properties. Are babies sensitive to the number of sounds, say the beats of a drum? Prentice Starkey's team used a virtuoso version of the habituation technique. They habituated babies to pictures of two dots or three dots, then showed them a black disk that emitted either two or three drumbeats. If the babies had been habituated to two dots they would stare longer at the black disk when it emitted two beats; if habituated to three dots, they would stare longer when it emitted three beats. For the babies to behave like this, they would have to have formed a mental representation that puts the *number* of dots and the number of beats into the same category, a category that is independent of whether the things were seen or heard.

It seems as though anything that a baby can think of as separate entities it can enumerate. Its behaviour can be guided by the number of things that it experiences independently of what those things are. It is born with the ability to form a representation of the numerosity of a collection of things, and because its behaviour changes when the number changes, it can also tell whether a new collection has the same numerosity as a previous collection. This in turn implies that it can store in its memory and retrieve, at least in the short term, the numerosity of the previous collection.

Is there an upper limit to the baby's concept of numerosity? Can it enumerate 4 or 10 or 100? The maximum seems to be 3, though in some tasks the baby can distinguish 'more than 3' from 3, and it has been claimed that in certain circumstances babies can distinguish 5 from 4. However, we cannot be sure that this limitation lies in the baby's idea of numerosity rather than in its ability to perceive and to remember what has been perceived. Our understanding that numerosities have no limit seems to depend on our sense that it is always possible to keep adding one. Thus any limitation on the infant's part may have more to do with its ability to carry out successive additions, and the chain of reasoning needed to get from that to the idea that numbers have no upper limit.

The most likely limitation is the ability to take in the numerosity of a visual array of objects at a glance, and without counting. Even in adults, the limit is about 4. This seems to be a specialized process in visual perception,[10] which is usually given the name *subitizing*. Stanislas Dehaene, a psychologist, and Jean-Pierre Changeux, a neuroscientist, working in Paris, have created a computer model of this process which very simply and effectively extracts the number of objects from a visual display, disregarding their size, shape, or location. The representation that is extracted can then be trained to make comparisons. It is tempting to think that something like this has been built into the visual processing system of the infant's brain.[11]

INFANT ARITHMETIC

Possessing a concept of numerosity implies more than just being able to decide whether two collections do or do not have the same numerosity. It implies an ability to detect a change in numerosity when new members are added to the collection, or old members taken away. Are babies born with the ability to do this? How can we tell? We certainly cannot ask them: not only can babies not speak, they cannot understand speech either. This lack may not prevent babies from having *arithmetical expectations* based on their concept of numerosity. The problem is to discover whether they do.

Imagine now that you see me put one doll into a box you know to be empty, and then I add another. You will *expect* to find two dolls in the box. This expectation will be based on your knowledge of arithmetic, among other things (such as the fact that objects do not just appear or disappear). If I show you that there is just one doll in the box, your expectations will be violated and you might stare longer into the box, wondering what could have happened to the other doll.[12] Similarly, if you see me put two dolls into a previously empty box and then remove one, you will expect there to be one doll left. Again, this will be an arithmetical expectation. If you now find two dolls in the box this expectation will be violated, and you may again show some surprise. The question now can be framed as follows: will a baby have the same expectations as you? Will it expect adding one doll to another doll to produce two dolls and show surprise if it does not? Will the baby expect taking one doll away from two dolls to leave exactly one doll, and show surprise if it does not?

This is what Karen Wynn set out to test in her Infant Cognition Laboratory at the University of Arizona.[13] She made use of the fact that babies look longer at events that violate their expectations; to discover whether babies had arithmetical expectations turned out to be really very simple. Babies of 4 to 5 months were seated facing a small stage. In the addition experiment, they would see one Mickey Mouse doll on the stage; then a screen would come up concealing the doll. A hand would then appear holding a second doll, which would be placed behind the screen. The screen would come down revealing either two dolls, which is what the babies would expect if they had addition expectations, or one doll, which is what they would not expect. Did they look longer at the single doll? Indeed they did. Perhaps, the sceptical reader is already asking, one doll was for some reason more interesting for the babies to look at, and that's why they looked at it longer.

Wynn had two ways of countering this potential criticism. First, she pre-tested the babies by measuring their looking times at one doll and at two dolls. No difference. Second, she carried out a subtraction experiment. This time the babies looked at a stage with two dolls on it. The screen came up, but now a hand went behind the screen and could be seen to remove a doll. The screen came down to reveal one doll, which is in line with a subtraction expectation, or two dolls, which is not. Would the babies now look longer at two dolls? Indeed they did, suggesting that they expected that two dolls, take away one doll, would leave just one doll.

Now we have a nice control for which display the babies might have preferred without any arithmetical expectations at all. A display of one doll was expected in the subtraction experiment but not in the addition experiment; so, if the babies did have arithmetical expectations, they should look longer at the same display after seeing the subtraction occur. This is just what happened. Similarly, a display of two dolls would be expected in the addition experiment but not the subtraction, and again the babies looked longer at the same display when it followed an arithmetical process that makes it unexpected.

This result has been replicated by Tony Simon at the Georgia Institute of Technology and his co-workers Susan Hespos and Philippe Rochat at Emory University.[14] They introduced a new twist to the experiment. Will babies (this time from 3 to 5 months) actually give more weight to arithmetical expectations than to other kinds of expectations about objects? In their experiment, the babies saw two kinds of doll—Elmo and Ernie, from the TV show 'Sesame Street', which are different in shape and colour. This allowed the experimenters to violate expectations about the identity of the object—Elmo surreptitiously changing into Ernie—as well as arithmetical expectations. They used 'possible' and 'impossible' trials in the addition experiment in which the baby saw one object, then the screen came up and they saw a hand put a second object behind the screen (Table 3.1).

The remarkable finding was that the babies responded very strongly when the outcome was arithmetically impossible, but did not seem to care at all when Elmos changed into Ernies. It is beyond all question that babies of this age are well able to discriminate shapes and colours and they do have expectations about objects appearing or disappearing without obvious reason, and about at least some of their physical properties. This seems to me good evidence not only that babies are born with the capacity to form arithmetical expectations of this simple sort, but that it is actually more important to them than some other, more obvious types of expectation.

Babies, as we have seen, have a sense of numerosity and can carry out a process of adding or subtracting one from small numerosities to develop accurate arithmetical expectations. We do not know whether babies' arithmetical expectations are really general, as ours are. For

Possible outcomes	*Impossible outcomes*
Elmo + Elmo = 2 Elmos	Elmo + Elmo = Elmo (impossible arithmetic)
Ernie + Ernie = 2 Ernies	Elmo + Elmo = Elmo + Ernie (impossible identity)
Elmo + Ernie = Elmo and Ernie	Elmo + Elmo = Ernie (impossible arithmetic and identity)

Table 3.1 Outcomes of trials with Elmo and Ernie.

example, do they have a general expectation that whenever you take an object from a collection you are left with one fewer? Do they even have expectations that apply to all collections of a given numerosity, that whenever you take 1 away from a collection of 2 you will be left with 1? They probably have this latter expectation, but we cannot be sure. We cannot even be sure that they have in mind the numerosities that they can distinguish in these experiments, in the sense that they can think about them in the absence of a collection with that numerosity, or whether indeed they can think about them at all.

We believe that if you take one thing away from a collection, then the collection will be smaller, it will have fewer members. It is by no means clear that babies understand this just because they notice a *difference* from their expectations when you take one doll away from two dolls. Similarly, we believe that adding one increases the numerosity of a collection, and again all we know about babies is that they notice the difference from their expectations.

Where babies have arithmetical expectations, we as adults also have arithmetical *beliefs* which we can formulate explicitly if we have to. Now, babies may also have beliefs about adding and subtracting, but it is very difficult for us to find out because the best evidence—explicit formulation in words—depends on speech. Nevertheless, babies' behaviour does appear to be constrained by the truths of arithmetic, even if they do not actually believe that if two objects are presented and just one of them is removed, then exactly one of them will be left. In this sense, they do seem to possess a kind of innate arithmetic.[15]

The reason why learning to count is so important is that it helps the baby build on these innate capacities so that it understands that numerosities have a size sequence, so that it can bring to mind individual numerosities, and so that it can have entirely explicit beliefs about the relationship between numbers and the effects of operations on numerosities.

Learning to Count—Some Simple Ways

Counting is basic. It makes the first bridge from the infant's innate capacity for numerosity to the more advanced mathematical achievements of the culture into which it was born. The least mathematical of cultures enable their members to do much more than the infant can. The members can keep track of quite large numerosities by counting with special number words or body-part names; they can do arithmetic beyond adding or subtracting one from small numerosities which they will need for trading or for ritual exchanges.

As any adult knows, counting is one of the easiest things to do. So why, if children are born to count, does it take so long for them to learn to do it? They start around two years old, and may be more than six years old before they have a good grasp of how to count and how to use counting.

Counting, it turns out, is not as simple as it first seems. Let us reflect for a moment on what practical skills we need to have mastered to count a collection of five toy dinosaurs. First, we need to know the number words from 'one' to 'five' (or, more generally, we need to know five counting words that we always keep in the same order). Second, we need to link each of these words with one and only one object: no word must be used more than once and all objects must be counted. That is, we must put each object in *one-to-one correspondence* with the counting words. Third, we must be in a position to announce the number of toy dinosaurs by using the last counting word used: 'One, two, three, four, five. Five toy dinosaurs.'[16]

Learning the sequence of counting words is the first way in which children connect their innate concept of numerosity with the cultural practices of the society into which they are born. As we saw in Chapter 2, not all societies use specialized words for counting. Many use the names of body parts. But all use the words in a fixed, unalterable sequence. You could count with the word sequence 'one, seven, five' so that each time 3 objects were counted, 'five' was announced. In this case the word 'five' would just mean 3. Similarly, you could count 'thumb, wrist, shoulder' every time so that the word 'shoulder' meant 3. What you cannot do is change the order, for then 'five' or 'shoulder' would have different meanings each time they were used and no one could understand how many had been counted.

Even learning the sequence of number words is not that straightforward. Children of two or three years often think of the first few number words as just one big word—'onetwothreefourfive'—and it takes them some time to learn that this big word is really five small words.[17] Of course, if there's just one big word, they cannot put the sequence into one-to-one correspondence with objects to be counted. It also takes a while for children to get the word sequence correct:

> A child of 3½ trying to count eight objects: 'One, two, three, four, eight, ten eleben. No, try dat again. One, two, three, four, five, ten, eleben. No, try dat again. One! two! three-ee-four, five, ten, eleben. No. . . . [finally] . . . One, two, four, five, six seven, eleven! Whew!'[18]

But knowing that there is a fixed sequence of separate words is not sufficient for knowing that these words are used for counting, that they have a role in finding the numerosity of a collection. As I pointed out at length in Chapter 1, number words have many meanings, not just a numerosity meaning. The child therefore has to separate counting uses from other uses which may be much more frequent around the house or school: telling the time, measuring, putting things in order, classroom numbers, TV channel numbers, house numbers, and so on. This suggests (though there is no direct evidence) that the acquisition of the word sequence and its use in counting will depend heavily on how it is taught and on the contexts in which it is learned.[19]

One-to-one correspondence appears at about two years of age quite independently of learning the sequence of counting words. At two, children happily give one sweet to each person, put one cup with each saucer, and can name each person in a room or a picture—or point to them—once and only once: 'Daddy, Mummy, Amy, Auntie Carole, me!'[20] If you show a 'puppet who is not very good at counting' counting the same object twice or missing an object altogether, children of three-and-a-half are very good at spotting these violations of one-to-one correspondence.[21] And almost all children point to each object when they count, even when they can use the number words correctly, so there is one-to-one correspondence between objects, points, and words.[22]

Children of three-and-a-half are also proficient at giving the last word in the count as the number of objects counted. Rochel Gelman, a psychologist at the University of California at Los Angeles (UCLA), whose work on children's number abilities has been the most influential since Piaget, calls this the cardinal word principle.[23] She, and others, have pointed out that merely doing this doesn't mean that children really know that the last number spoken is the numerosity of the set. They could just be imitating an adult routine. We can ask children of this age to count a collection of toy dinosaurs, and they can reply, correctly, 'One, two, three, four.' If we then ask them, 'How many toy dinosaurs are there?' they may well go back and count them again. This may be because they don't understand that the process of counting will give them the numerosity of things counted. On the other hand, given that they've just counted how many, they may think that the adult has detected an error and is asking them to do it again.

Some children of this age, though they may count in some circumstances, do not always see the point of it. If you ask them to give you three toy dinosaurs, they may just grab a handful and give them to you without counting. Karen Wynn[24] calls them 'grabbers'. Although they do use the last word of a count to say how many, grabbers often have not yet grasped the role of number words in counting—sometimes they think that the number word is just a label that attaches to an object. Here is what Adam, a grabber, did in one of Wynn's tasks:

> EXPERIMENTER (E): So how many are there?
>
> ADAM (A) (Counting three objects): One, two, five!
>
> E (Pointing towards the three items): So there's five here?
>
> A: No, that's *five* (pointing to the item he'd tagged 'five').
>
> . . .
>
> E: What if you counted this way, one, two, five? (Experimenter counts the objects in a different order than Adam has been doing.)
>
> A: No, *this* is five (pointing to the one he has consistently tagged 'five').

Other children, whom Wynn calls 'counters', usually a few months older, will count, either aloud or silently, passing you the toys one by one. They also reliably give you the last word of the count in answer to 'How many?' These children are initially able to count only small numerosities, and probably build up their competence systematically from 1 to 2, from 2 to 3, from 3 to 4, and so on. In a give-a-number task, they will start by being able reliably to give 1, then to give 2, but perhaps not 3, then 3 but perhaps not 4. So by three-and-a-half most children have a grasp of small numerosities, and know that counting is a way to find the numerosity of a collection.

Children also need to understand that it doesn't matter in which order they count the objects in a collection, or indeed what those objects are. Gelman calls these the 'order-irrelevance principle' and the 'abstractness principle'.[25] It is, however, true that even when they normally obey these principles, they are still better at counting solid objects than actions or sounds, especially objects they can actually move about,[26] and also do better when the objects are lined up in a row and they start at one end rather than in the middle. It's generally believed that these differences in performance are due not to a weakness in the basic ideas, but to other problems in carrying out the task, including the effects of practice. For the same reason they are better at counting with small numbers than large. They have much less practice with large numbers: who would teach their child 'one hundred and seventy-three' before they teach 'three'?

Gelman's three 'how-to-count principles'—cardinal word, order-irrelevance, and abstractness—along with a fourth principle for one-to-one correspondence guide the acquisition of verbal counting skills. It is clear that a grasp of the principles follows from understanding the concept of numerosity. Collections are not intrinsically ordered. Understanding this means that you understand the order-irrelevancy principle. There is also no constraint on the kinds of things that can be members of a collection, provided they can be individuated. Understanding this implies holding the principle of abstractness. Of course, children, and adults, may possess the concept of numerosity without fully understanding and without having derived all the principles that validly follow from it.

The how-to-count principles for putting counting words to members of a collection are complex, since counting means being able to link each member of a collection, once and only once, to a fixed sequence of words. In this way, the further you count in the sequence the larger the numerosity you have counted. If you count from the word 'one' to the word 'five' in the usual English sequence, then the members counted at the word 'four' are a sub-collection of the final collection of five objects; similarly, the members counted at 'three' are a sub-collection of the collections of four and five.

Specifically, the cardinal word principle—the last number named in a count is the numerosity of the collection counted—also follows from the concept of numerosity, since you are establishing a correlation between members of a collection whose numerosity you do know, the number words up to five, say, and members of the collection of things to be counted, whose numerosity you do not know. It may follow in a practical way as well. Recall that infants can recognize the numerosities of objects up to about 3. In children and adults this ability to take in the numerosity of a visual array of objects at a glance, and without counting, is known as *subitizing*.

Karen Fuson, from Northwestern University in Illinois, who carries out her research in some of the toughest schools and homes in Chicago's notorious South Side, suggests that children may notice that when they count a collection 'one, two, three', they get the same number as when they subitize the collection. This helps them to see that counting up to N is a way of establishing that a collection has N objects in it.[27]

Repeating the count, and getting the same number as was obtained from subitizing, will reinforce the idea that every number name represents a unique numerosity. Again, this is something obvious to us adults, but it may not be obvious to children, especially as in practice children will sometimes count the same collection and get different results. They will count (or miscount) 'one, two, three dinosaurs', and may count again 'one, two, four dinosaurs', and then again 'one two three four dinosaurs'. They may wonder whether different number words can name the same numerosity, the numerosity of the collection of dinosaurs. Perhaps they will ask themselves: 'Does counting always give me the numerosity I get from subitizing?' (though they are unlikely to frame the question in exactly those words).

One of the child's real problems at around this age, about four years old, is, in my view, the conflict that can occur between the numerosities they achieve by counting and the numerosities provided by their innate ability to subitize small collections of objects. Suppose that subitizing yields four objects but counting comes up with five; which are they to believe? Will they suspend judgement and recount? At the moment, little is known about the causes and consequences of this conflict.

The child will have still other ways of estimating the numerosity of a collection. In general, they will have noticed that more things take up more room than fewer things, so the size of the array will be a useful clue. Packing density will also be a clue to numerosity. Piaget was among the first to see that a full grasp of the concept of numerosity meant being able to abstract away from—or ignore—these superficial clues, so that you do not think, for example, that there are more things just because they are more spread out (or more closely packed together). He saw the development of the child's thinking in general as a move away from the particular to the general and abstract (indeed he thought that really abstract formal reasoning doesn't emerge until puberty—which may not correspond to the intuitions of parents with teenage children[28]).

Children also use one-to-one correspondence to establish which of two sets has more things in it. Even children as young as four can give two people the same number of sweets by using a 'one for Bill, one for Mary' strategy. In one experiment, the experimenter counted out Bill's share, and then asked the children who had successfully shared out the sweets how many sweets Mary had, but less than half of them spontaneously made the inference that Mary had the same number as Bill. Most tried to count out Mary's share. So the match between *these* two ways of determining numerosity—one-to-one correspondence and counting—is by no means clear to children at this stage.[29]

The conflict between the innate systems and counting becomes most surprising when children actually change their mind about the numerical relationship between the collections. Piaget observed that moving objects around, without adding or subtracting an object, will make children as old as five or six think that the numerosity has changed. He would lay out two lines of coins, one above the other, so that each coin in one line was clearly paired with a coin in the other. The children under these conditions had no trouble saying whether or not the two lines had the same number of coins. Then, in full view of the children, Piaget spread out the coins in one line so that the line was longer than the other line. It was clear that he had added none and taken none away. He asked the children which line had more coins, and they said the longer line.

The conflict between the different sources of evidence—different 'cues'—for the numerosities can be seen very clearly in the way children between four and six try to establish whether two sets have the same number. What seems to happen is that during this period, they come to suppress perceptual cues such as the spacing of objects, and to depend exclusively on genuine numerosity information, such as correspondence and counting. They cease to be fooled by changing the spacing of objects. In Piagetian terms, number is 'conserved' under perceptual transformations. There are three stages in the child's progress to conservation.

Piagetian Stages in Achieving 'Conservation of Number'[30]
(The basic situation: six little bottles, about one inch high, the kind used in dolls' games, are put on the table, and the child is shown a set of glasses on a tray.) Look at these little bottles. What shall we need if we want to drink? *Glasses*. Well, there they are. Take off the tray just enough glasses, the same number as there are bottles, one for each bottle. (*Child's responses in italics*.)

Stage I: No correspondence or equivalence. Here the child seems to rely solely on perceptual cues like spacing, and doesn't count or use one-to-one correspondence.

Car (5 years 2 months) arranged them so that each bottle had its glass. (He had taken all the glasses, so he removed some and left 5. He tried to make these correspond to the 6 bottles by spacing them out so as to make a row the same length.) Is there the same number of glasses and bottles? *Yes*. Exactly? *Yes*. (The bottles were then moved closer together so that the two rows were no longer the same length.) Are they the same? *No*. Why? *There aren't many bottles*. Are there more glasses or more bottles? *More glasses* (pushing them a little closer together). Is that the same number of glasses and bottles now? *Yes*. Why did you do that? *Because that makes them less*.

Stage II: Children can do one-to-one correspondence, but still rely more on perceptual cues. In Piaget's terms, they haven't yet achieved 'conservation of number' under perceptual transformations.

Mog (4; 4) estimated that he needed 9 glasses for the 6 bottles, then made one-one correspondence and removed the 3 that were left over, and said spontaneously: *No, it wasn't the right number*. And are they the same now? *Yes*. (The glasses were put closer together and the bottles spread out a little.) Is there the same number of glasses and bottles? *No*. Where are there more? *There are more bottles*.

Children can be quite explicit about the evidence they are relying on to make a judgement. Counting can override perceptual cues at this stage.

Gal (5; 1) made 6 glasses correspond to 6 bottles. The glasses were then grouped together. Is there the same number of glasses and bottles? *No, it's bigger there* (the bottles) *and smaller here* (the glasses). (The bottles were then grouped together and the glasses spread out.) *Now there are more glasses*. Why? *Because the bottles are close together and the glasses are all spread out*. Count the glasses. *1, 2, . . . 6*. Count the bottles. *1, 2, . . . 6*. They're the same then? *Yes*. What made you say they weren't the same? *It was because the bottles are very small*.

Stage III: Children have achieved conservation of number. They now rely on one-to-one correspondence or counting and are not deceived by changing the preceptual cues.

Lau (6; 2) made 6 glasses correspond to 6 bottles. The glasses were then grouped together. Are they still the same? *Yes, it's the same number of glasses. You've only put them close together, but it's still the same number*. And now, are there more bottles (grouped) or glasses (spaced out)? *They're still the same. You've only put the bottles close together*.

Piaget believed that counting, and learning number words to do it, was not important, and certainly not necessary, for constructing the concept of numerosity, which he thought was built up from logical concepts and reasoning, around the ages of four to six, until possession of the concept was demonstrated by conservation of number under transformations. Following Piaget, the French virtually outlawed teaching counting in nursery schools: clearly there would be no point if the child has first to develop the Piagetian prerequisites of transitive inference, class inclusion, and so on.[31]

Counting Larger Numbers: Why It's Easier in Chinese

The conceptual tools provided by the culture are not all the same, even for something as basic as counting. In Chapter 2 we saw that there were lots of different counting systems, some using body-part names; some with bases of 20 or 60; others, like Ainu, which used overcounting and subtraction; others, like ours, that used only addition; some with big numbers always first, like our numeral system; others with the smaller number first, as in German *ein-und-zwanzig* ('one-and-twenty'). The child will have different problems to solve in trying to master these local counting systems. Do some of them make life harder for the child, or are they all really much the same? Unfortunately, research in this area is just beginning, but there is one feature of counting systems that we now know makes life harder: irregularity.

The innate structure of our representations of numerosity is not in base-10 form. It may work only for numbers less than 10: the evidence from infants has shown abilities only up to about 4. Even if nature has endowed us with a way of constructing mental representations of numbers as large as we choose, there is no reason to suppose that it will be in base 10. However, English-speaking children need to learn the local cultural practice of counting in tens. In fact, they have to learn two quite different principles of counting in tens. The first learned is the verbal system, which is not a place-value system but an enciphered or 'name-value' system. We have special names for (some but not all of) the powers of ten: 'ten', 'hundred', 'thousand', 'million'. We say 'two hundred and twelve', not 'two one two' (which is how the poet-astronomers of the fifth-century Indus Valley civilization did it, as we saw in Chapter 2). We also have to learn the numeral system, which is indeed place-value. And we have to learn how to 'transcode', to use the technical term, from one to the other. The most basic thing that children have to learn is that larger numbers are composed of smaller numbers, that both systems use 'additive composition'. That is, they have to learn that *twelve* and *12* both mean ten plus two.

Now, different languages can make the additive composition of the number-word system clear or obscure. In English, we count from

1 to 9, with each number represented by a single word. *Ten* is a word, so are *eleven*, *twelve*, and so on up to *twenty*. In Chinese, each number up to 10 is also represented by a single word, but thereafter things are regular (Table 3.2). For Chinese children there is no trouble with the teens, as they do not have to learn special words for them. But how is the English-speaking child to know that *eleven* is really ten plus one, or *twelve* ten plus two? They sound like single words, and are written as single words. Some children pick up on the fact that there is a new, teen, series after ten. This can lead to mistakes such as over-regularizing from the later teens to the earlier. Some children briefly count *ten, eleventeen, twelveteen*. The first decade is irregular not only in being single words—other decades are two words—and not only in having the first two words not sounding at all like teens; but those that have the word teen in them have teen in the wrong place—after the unit number ('thirteen' is a 3, 10), rather than before for all other decades in English ('twenty-three' is 20, 3).

Our children have to learn that *-ty* is the bit of the word that stands for 10, and that *twen-*, *thir-*, and *fif-* (none of which are words in English) mean two, three, and five (or twice, thrice, and five times). Chinese children, on the other hand, need to learn no new words or new bits of words for the decades: the number of tens is explicit—*two ten, three ten*, and so on. When it comes to representing units, the unit number always comes straight after the ten, unlike in French, where decades plus one are different from decades plus a higher number—*vingt-et-un*, but *vingt-deux*. Does this simplicity really give Chinese children an advantage in learning their numbers? Do they learn the principle of additive composition more easily?

Chinese-speaking Taiwanese children of 4, 5, and 6 years old turn out to be spectacularly better at counting than their US counterparts. The US children were far worse when they got to the teens, while the Taiwanese children scarcely made any mistakes.[32] What is more, Taiwanese children understand how tens and units are added together in the base-10 system that is common to Chinese and English; in fact they seem to understand how to add numbers together to make a target number far better than US children do. This was shown recently by a study by Peter Bryant with six-year-olds. He used a simulated shop where things cost 6p or 11p, and children

1	*yi*	20	*er shi*
2	*er*	21	*er shi yi*
3	*san*	22	*er shi er*
4	*si*	23	*er shi san*
5	*wu*	24	*er shi si*
6	*liu*	25	*er shi wu*
7	*qi*	26	*er shi liu*
8	*ba*	27	*er shi qi*
9	*jiu*	28	*er shi ba*
10	*shi*	29	*er shi jiu*
11	*shi yi*	30	*san shi*
12	*shi er*		
13	*shi san*	100	*yi bai*
14	*shi si*		
15	*shi wu*	200	*er bai*
16	*shi liu*		
17	*shi qi*	231	*er bai san shi yi*
18	*shi ba*		
19	*shi jiu*		

Table 3.2 Chinese number words.

had 1p, 5p, and 10p coins (or their Chinese equivalents). Both groups were equally good at counting out the 1 units, but the Chinese children were much better at paying with 10 + 1 rather than eleven 1's, which is what one would expect if the transparency of the Chinese verbal system makes additive composition so much easier to understand. What is more, they seemed to understand the principle better, since they transferred this to paying 5 + 1 for the 6-unit purchase.[33]

Even when children understand that their language uses a base-10 representation, there is still the conflict between the way the base is encoded in the name-value system of the words and in the place-value system of the numerals. Children learn that *ten* is encoded as 10, *twenty* as 20, *one hundred* as 100, *two hundred* as 200. They will also have learned about additive composition. So *one hundred* plus *one* can be formed simply by sticking the two parts together (with an *and* in between) as *one hundred and one; one hundred* plus *forty-seven* is *one hundred and forty-seven*. It is perhaps not surprising that in the early stages of learning to write numerals, English children write 'one hundred and one' as 1001, or 'one hundred and twenty-nine' as '10029'.[34] An explanation for this was proposed by Richard Power and Maria dal Martello, from the University of Padua in Italy, who had studied the writing errors of Italian children aged six and seven.[35] They tested the children's ability to write numbers to dictation. The children did fine with two-digit numbers, like 27, and with multiples of powers of 10, such as 100 or 200, but many were still making mistakes writing three-digit numbers like *due cento venti sette* and *tre cento cinque*. Instead of writing 227 and 305, they wrote '20027' and '3005', just like the English children. They were writing the name values 200 and 300, and using additive composition to stick on the other part of the number. But why didn't they write *venti sette* as 207? In a previous study with one of the founding fathers of artificial intelligence, Christopher Longuet-Higgins, Power had figured out the few general rules that relate number words to numerals in a very wide range of languages. In fact, they wrote a computer program that would work out the relationship for any language, no matter which base system it used, given just a few examples. Two of these rules seemed to have a direct application here. One they called 'concatenation'. For two-digit numbers, the child need only take the 'major term', the multiple of the tens (2), and the 'minor term', the multiple of the units (7), and concatenate them to get the correct way of writing 27. However, for three-digit numbers this doesn't work. Children need to extract another rule from the examples they hear and see, which Power and dal Martello called 'overwriting from the right'. For *two hundred and seven* the rightmost 0 must be overwritten by the 7; for *two hundred and twenty seven* the two rightmost 0's must be overwritten by the 27. This is a more complicated rule than concatenation, and since children learn three-digit numbers after they have learned two-digit numbers, it will be acquired later. So there will be a stage when they have the easier concatenation rule but are still working on the overwriting rule.

In learning about numbers larger than those readily grasped by the innate Number Module, the child has to confront the conflict between two conflicting representational principles: name value in words and place value in numerals. This makes learning difficult, especially, as we have seen, where the name-value system is irregular, as it is in English and other European languages.

In Figure 3.2 I summarize how children add to their conceptual toolkit, by using body parts such as fingers, number words, and numerals. One of the problems they face is how to coordinate these different systems.

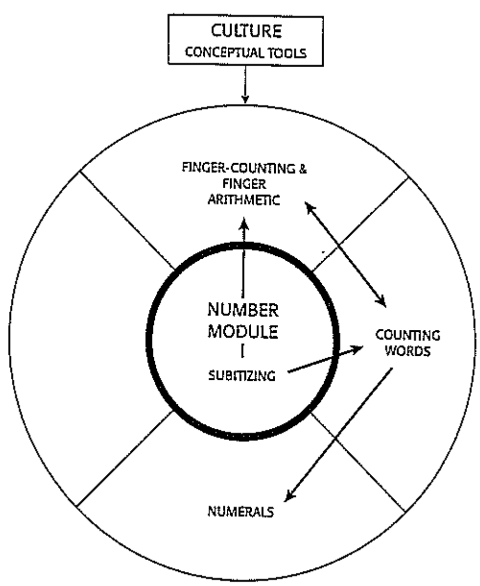

Figure 3.2 The child at about four years uses subitizing to support learning what the number words mean and how many fingers are held up. Numerals are learned primarily in connection with the number words.

ARE THERE SEX DIFFERENCES IN NUMBER ABILITIES?

Is there something about the female brain that makes it, on average, worse at mathematics than the male brain? This is an important question for our enquiry because if numerical capacity is sex-linked then this is positive evidence for its being coded in the genes. The development of primary sexual characteristics is of course sex-linked and coded in the genes; so is average brain size: human females everywhere have brains 20% smaller than males—1,200 cm^3 versus 1,500 cm^3.

However, it has been very difficult to show that differences in brain size have any cognitive consequences at all. There is no difference in average IQ. In academic achievement, boys have in the past outperformed girls by the age of eighteen. However, girls in England now outperform boys in all subjects at all ages. There is one exception to this general rule: mathematics. Girls are still doing worse at mathematics than boys (but read on).

This has been an official worry since Dr Cockcroft and his Committee of Inquiry into the Teaching of Mathematics produced his report for the British government. But that was in 1982. Before then, few seemed to care, and many thought it almost improper for girls to be good at maths. In a relatively enlightened *Handbook for Teachers*, issued in 1937, the British government advised:

> In mental capacity and intellectual interests [boys and girls] have much in common, the range of difference in either sex being greater than the difference between the sexes. But in early adolescence the thoughts of boys and girls are turning so strongly towards their future roles as men and women that it would be entirely inappropriate to base their education solely on their intellectual similarity.[36]

A father, withdrawing his daughters from Cheltenham Ladies' College when it introduced 'arithmetic' into the curriculum, wrote to Dorothea Beale, the headmistress, 'My dear lady, if my daughters were going to be bankers, it would be very well to teach arithmetic as you do, but really there is no need.' From the middle of the nineteenth century, when female education became an issue, until the Cockcroft report, the curriculum was designed to reflect its usefulness to boys and girls in their adult lives. Differences in mathematical performance were generally attributed to lack of interest on the part of girls, who would be mainly concerned with their future roles as wives and mothers. In 1923, the Board of Education attributed 'girls' inferiority' in mathematics partly to 'an impression among parents, which has influence on the timetable, that mathematics is unsuitable for girls'. Maths was frequently sacrificed to needlework. The Board even suggested that girls' poor performance could be due 'partly to unskilful teaching of an old-fashioned kind'.[37] This is scarcely surprising since very few females wishing to teach maths had had the opportunity to go to university, and fewer still to follow a graduate course in mathematics.

In many countries before the Second World War, including Britain and in many ways the more educationally progressive Austria and Germany, women were strongly discouraged or even prevented from doing mathematics at university. Margaret Wertheim, in her fascinating history of female exclusion from the 'priesthood' of mathematical physics, notes that in the oldest of all mathematical priesthoods, Pythagoras' community in Croton (southern Italy), women were full members, *mathematikoi*. There were even women teachers of mathematics, including Theano, Pythagoras' wife. This was not to last. In the ancient world there was a fundamental division between the male realm and the female realm, the Sky Father and the Earth Mother. For the later Pythagoreans this became a division between the realm of the psyche, which was male, and the realm of the body, which was female. Numbers were in the realm of the psyche, and so naturally a male pursuit, and women came to be excluded from its study. With very few exceptions, women were excluded from mathematics until very recently indeed.

Emmy Noether was one of the team that David Hilbert, perhaps the greatest mathematician of his time, assembled in Göttingen to help Einstein find the right mathematics for the relativistic theory of gravity. She made fundamental contributions to algebra, and to the search for a unified theory of general relativity and quantum mechanics. Yet her life illustrates what obstacles even the most talented of mathematicians had to overcome if they were women.

Despite being the daughter of a professor of mathematics, she was denied entry to university, and spent three years at teacher training college. She then was allowed to attend classes in mathematics at Erlangen, her father's university, but only as an 'auditor'. It was not until five years later that she was allowed formally to enrol. Her doctoral thesis was described as 'an awe-inspiring piece of work' by Hermann Weyl, one of the most celebrated mathematicians of his day. However, as Wertheim puts it, 'It was one thing for a woman to be educated in Germany; it was another matter for her to be employed'.[38] For the whole of her working life in Europe she never received a proper salary, even when she was finally offered an official position at Göttingen.

In the first half of this century, most girls in Europe and the USA harbouring the hope of going on to university were attending single-sex schools where they were taught by female teachers. With these obstacles to getting a degree in mathematics, there were very few properly qualified teachers of mathematics. There is no doubt that this put girls at a disadvantage. Today, when presumably girls and boys have more or less equivalent teaching, have the girls caught up?

In a provocative review, David Geary, a psychologist from the University of Missouri, assembled evidence from a wide range of industrialized countries to show that boys, on average, still outperform girls in mathematical problem-solving.[39] Among US teenagers, there are more boys than girls in the upper reaches of the SAT-M (Scholastic Aptitude Test—Mathematics), a requirement for university admission. The difference between boys and girls gets larger higher up the range.

Geary attributes the male advantage to their superior visuo-spatial skills, needed in the competition for mates when man was a hunter and woman a gatherer. These visuo-spatial skills, tempered in habitat navigation, will be evident in geometry, but also in solving non-geometrical problems formulated in words. One problem with this theory is that even in the USA at 17 years the *average* difference between boys and girls is still only 1%. The most recent cross-national comparisons using the same tests in all countries, the Third International Maths and Science Survey (TIMSS),[40] reinforces the overall picture that in most countries, including the USA, boys do not outperform girls at all (Table 3.3). However, there are still a few countries in which boys significantly outperform girls, most dramatically in

Age nine to ten			Age fourteen		
Country	*Mean (points)*	*Difference in favour of boys*	*Country*	*Mean (points)*	*Difference in favour of boys*
Singapore	625	−10	Singapore	601	0
Scotland	520	0	Hungary	502	1
USA	544	2	Canada	494	2
Canada	533	3	Germany	485	2
Hungary	549	5	Scotland	464	3
England	513	5	USA	476	5
Norway	502	5	Sweden	478	5
Japan	693	8*	France	493	8
Netherlands	577	15*	Japan	571	11*
			Switzerland	506	14*
			England	476	17*

*Statistically significant difference.

Table 3.3 International comparisons of sex differences at two ages in points.[41]

England. Interestingly, the gap between boys and girls gets bigger with age, indicative of the critical influence of educational practices. What is certainly clear from the TIMSS data is that the differences between countries, between educational practices, has a vastly greater effect on performance than the difference between the sexes.

TIMSS provides a snapshot of the situation three years ago, but in the most recent public examination figures in England, January 1997 at the time of writing, at the age of sixteen girls are outperforming boys in every subject, including mathematics. In some educational authorities girls are, overall, outperforming boys by nearly 50%.[42] So the urgent question is no longer why girls are performing less well than boys, but why teenage boys are doing so badly! No one has yet suggested that boys are inherently less intelligent than girls, nor that they might be better fitted to manual work than to brain work.

Geary refers to numerosity recognition and ordering, counting, and simple arithmetic as 'biologically primary abilities'.[43] These are precisely the abilities that seem to be functioning in infants, and, as we shall see in the next section, in our primate ancestors. Now, if there is to be a biological difference between males and females, here is where we should find it, since more advanced numerical skills will be deeply affected by educational opportunities. However, Geary admits he was unable to discover any evidence at all for sex differences in these biologically primary abilities.

The fact that there are no sex differences in these basic abilities, and probably no biologically based sex differences in higher abilities, does not mean that the basic abilities are not coded in our genes. It tells us only that they may not be sex-linked properties of the genome. So we shall have to look elsewhere for the genes.