THE UNIVERSAL HISTORY OF

NUMBERS

FROM PREHISTORY TO THE INVENTION OF THE COMPUTER

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Translated from the French
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CHAPTER 1

EXPLAINING THE ORIGINS

Ethnological and Psychological Approaches to the Sources of Numbers

WHEN THE SLATE WAS CLEAN

There must have been a time when nobody knew how to count. All we can surmise is that the concept of number must then have been indissociable from actual objects – nothing very much more than a direct apperception of the plurality of things. In this picture of early humanity, no one would have been able to conceive of a number as such, that is to say as an abstraction, nor to grasp the fact that sets such as "day-and-night", a brace of hares, the wings of a bird, or the eyes, ears, arms and legs of a human being had a common property, that of "being two".

Mathematics has made such rapid and spectacular progress in what are still relatively recent periods that we may find it hard to credit the existence of a time without number. However, research into behaviour in early infancy and ethnographic studies of contemporary so-called primitive populations support such a hypothesis.

CAN ANIMALS COUNT?

Some animal species possess some kind of notion of number. At a rudimentary level, they can distinguish concrete quantities (an ability that must be differentiated from the ability to count numbers in abstract). For want of a better term we will call animals' basic number-recognition the *sense of number*. It is a sense which human infants do not possess at birth.

Humans do not constitute the only species endowed with intelligence: the higher animals also have considerable problem-solving abilities. For example, hungry foxes have been seen to "play dead" so as to attract the crows they intend to eat. In Kenya, lions that previously hunted alone learned to hunt in a pack so as to chase prey towards a prepared ambush. Monkeys and other primates, of course, are not only able to make tools but also to learn how to manipulate non-verbal symbols. A much-quoted example of the first ability is that of the monkey who constructed a long bamboo tube so as to pick bananas that were out of reach. Chimpanzees have been taught to use tokens of different shapes to obtain bananas, grapes, water, and so on, and some even ended up hoarding the tokens against future needs. However, we must be careful not to be taken in by the

kind of "animal intelligence" that you can see at the circus and the fairground. Dogs that can "count" are examples of effective training or (more likely) of clever trickery, not of the intellectual properties of canine minds. However, there are some very interesting cases of number-sense in the animal world.

Domesticated animals (for instance, dogs, cats, monkeys, elephants) notice straight away if one item is missing from a small set of familiar objects. In some species, mothers show by their behaviour that they know if they are missing one or more than one of their litter. A sense of number is marginally present in such reactions. The animal possesses a natural disposition to recognise that a small set seen for a second ti undergone a numerical change.

Some birds have shown that they can be trained to recognis precise quantities. Goldfinches, when trained to choose between tw ent piles of seed, usually manage to distinguish successfully betwee and one, three and two, four and two, four and three, and six an

Even more striking is the untutored ability of nightingales, magp crows to distinguish between concrete sets ranging from one to t four. The story goes that a squire wanted to destroy a crow that ha its nest in his castle's watchtower. Each time he got near the nest, t flew off and waited on a nearby branch for the squire to give up down. One day the squire thought of a trick. He got two of his me into the tower. After a few minutes, one went down, but the othe behind. But the crow wasn't fooled, and waited for the second ma down too before coming back to his nest. Then they tried the tri three men in the tower, two of them going down: but the third ma wait as long as he liked, the crow knew that he was there. The pl worked when five or six men went up, showing that the crow could not discriminate between numbers greater than three or four.

These instances show that some animals have a potential which is more fully developed in humans. What we see in domesticated animals is a rudimentary perception of equivalence and non-equivalence between sets, but only in respect of numerically small sets. In goldfinches, there is something more than just a perception of equivalence – there seems to be a sense of "more than" and "less than". Once trained, these birds seem to have a perception of intensity, halfway between a perception of quantity (which requires an ability to numerate beyond a certain point) and a perception of quality. However, it only works for goldfinches when the "moreness" or "lessness" is quite large; the bird will almost always confuse five and four, seven and five, eight and six, ten and six. In other words, goldfinches can recognise differences of intensity if they are large enough, but not otherwise.

Crows have rather greater abilities: they can recognise equivalence and non-equivalence, they have considerable powers of memory, and they can perceive the relative magnitudes of two sets of the same kind separated in time and space. Obviously, crows do not count in the sense that we do, since in the absence of any generalising or abstracting capacity they cannot conceive of any "absolute quantity". But they do manage to distinguish concrete quantities. They do therefore seem to have a basic number-sense.

NUMBERS AND SMALL CHILDREN

Human infants have few innate abilities, but they do possess something that animals never have: a potential to assimilate and to recreate stage by stage the conquests of civilisation. This inherited potential is only brought out by the training and education that the child receives from the adults and other children in his or her environment. In the absence of permanent contact with a social milieu, this human potential remains undeveloped – as is shown by the numerous cases of *enfants sauvages*. (These are children brought up by or with animals in the wild, as in François Truffaut's film, *The Wild Child*. Of those recaptured, none ever learned to speak and most died in adolescence.)

We should not imagine a child as a miniature adult, lacking only judgement and knowledge. On the contrary, as child psychology has shown, children live in their own worlds, with distinct mentalities obeying their own specific laws. Adults cannot actually enter this world, cannot go back to their own beginnings. Our own childhood memories are illusions, reconstructions of the past based on adult ways of thinking.

But infancy is nonetheless the necessary prerequisite for the eventual transformation of the child into an adult. It is a long-drawn-out phase of preparation, in which the various stages in the development of human intelligence are re-enacted and reconstitute the successive steps through which our ancestors must have gone since the dawn of time.

According to N. Sillamy (1967), three main periods are distinguished: *infancy* (up to three years of age), *middle childhood* (from three to six or seven); and *late childhood*, which ends at puberty. However, a child's intellectual and emotional growth does not follow a steady and linear pattern. Piaget (1936) distinguishes five well-defined phases:

1. a *sensory-motor period* (up to two years of age) during which the child forms concepts of "object" out of fragmentary perceptions and the concept of "self" as distinct from others;
2. a *pre-operative stage* (from two to four years of age), characterised by egocentric and anthropomorphic ways of thinking ("look, mummy, the moon is following me!");
3. an *intuitive period* (from four to six), characterised by intellectual perceptions unaccompanied by reasoning; the child performs acts which he or she would be incapable of deducing, for example, pouring a liquid from one container into another of a different shape, whilst believing that the volume also changes;
4. a stage of *concrete operations* (from eight to twelve) in which, despite acquiring some operational concepts (such as class, series, number, causality), the child's thought-processes remain firmly bound to the concrete;
5. a period (around puberty) characterised by the emergence of *formal operations*, when the child becomes able to make hypotheses and test them, and to operate with abstract concepts.

Even more precisely: the new-born infant in the cradle perceives the world solely as variations of light and sound. Senses of touch, hearing and sight slowly grow more acute. From six to twelve months, the infant acquires some overall grasp of the space occupied by the things and people in its immediate environment. Little by little the child begins to make associations and to perceive differences and similarities. In this way the child forms representations of relatively simple groupings of beings and objects which are familiar both by nature and in number. At this age, therefore, the child is able to reassemble into one group a set of objects which have previously been moved apart. If one thing is missing from a familiar set of objects, the child immediately notices. But the abstraction of number – which the child simply feels, as if it were a feature of the objects themselves – is beyond the child's grasp. At this age babies do not use their fingers to indicate a number.

Between twelve and eighteen months, the infant progressively learns to distinguish between one, two and several objects, and to tell at a glance the relative sizes of two small collections of things. However, the infant's numerical capabilities still remain limited, to the extent that no clear distinction is made between the numbers and the collections that they represent. In other words, until the child has grasped the generic principle of the natural numbers ($2 = 1 + 1$; $3 = 2 + 1$; $4 = 3 + 1$, etc.), numbers remain nothing more than "number-groupings", not separable from the concrete nature of the items present, and they can only be recognised by the principle of *pairing* (for instance, on seeing two sets of objects lined up next to each other).

Oddly enough, when a child has acquired the use of speech and learned to name the first few numbers, he or she often has great difficulty in symbolising the number three. Children often count from one to two and then miss three, jumping straight to four. Although the child can recognise, visually and intuitively, the concrete quantities from one to four, at this stage of development he or she is still at the very doorstep of abstract numbering, which corresponds to *one, two, many*.

However, once this stage is passed (at between three and four years of age, according to Piaget), the child quickly becomes able to count properly. From then on, progress is made by virtue of the fact that the abstract concept of number progressively takes over from the purely perceptual aspect of a collection of objects. The road lies open which leads on to the acquisition of a true grasp of abstract calculation. For this reason, teachers call this phase the "pre-arithmetical stage" of intellectual development. The child will first learn to count up to ten, relying heavily on the use of fingers; then the number series is progressively extended as the capacity for abstraction increases.

ARITHMETIC AND THE BODY

The importance of the hand, and more generally of the body in children's acquisition of arithmetic can hardly be exaggerated. Inadequate access to or use of this "counting instrument" can cause serious learning difficulties.

> In earliest infancy, the child plays with his or her fingers. It constitutes the first notion of the child's own body. Then the child touches everything in order to make acquaintance with the world, and this also is done primarily with the hands. One day, a well-intentioned teacher who wanted arithmetic to be "mental", forbade finger-counting in his class. Without realising it, the teacher had denied the children the use of their bodies, and forbidden the association of mathematics with their bodies. I've seen many children profoundly relieved to be able to use their hands again: their bodies were at last accepted [. . .] Spatio-temporal disabilities can likewise make learning mathematics very difficult. Inadequate grasp of the notions of "higher than" and "lower than" affect the concepts of number, and all operations and relations between them. The unit digits are written to the right, and the hundred digits are written to the left, so a child who cannot tell left from right cannot write numbers properly or begin an operation at all easily. Number skills and the whole set of logical operations of arithmetic can thus be seriously undermined by failure to accept the body. [L. Weyl-Kailey (1985)]

NUMBERS AND THE PRIMITIVE MIND

A good number of so-called primitive people in the world today seem similarly unable to grasp number as an abstract concept. Amongst these populations, number is "felt" and "registered", but it is perceived as

a *quality*, rather as we perceive smell, colour, noise, or the presence of a person or thing outside of ourselves. In other words, "primitive" peoples are affected only by changes in their visual field, in a direct subject-object relationship. Their grasp of number is thus limited to what their predispositions allow them to see in a single visual glance.

However, that does not mean that they have no perception of quantity. It is just that the plurality of beings and things is measured by them not in a quantitative but in a qualitative way, without differentiating individual items. Cardinal reckoning of this sort is never fixed in the abstract, but always related to concrete sets, varying naturally according to the type of set considered.

> A well-defined and appropriately limited set of things or beings, provided it is of interest to the primitive observer, will be memorised with all its characteristics. In the primitive's mental representation of it, the exact number of the things or beings involved is implicit: it resembles a quality by which this set is different from another group consisting of one or several more or fewer members. Consequently, when he sets eyes on the set for a second time, the primitive knows if it is complete or if it is larger or smaller than it was previously. [L. Lévy-Bruhl (1928)]

ONE, TWO . . . MANY

In the first years of the twentieth century, there were several "primitive" peoples still at this basic stage of numbering: Bushmen (South Africa), Zulus (South and Central Africa), Pygmies (Central Africa), Botocudos (Brazil), Fuegians (South America), the Kamilarai and Aranda peoples in Australia, the natives of the Murray Islands, off Cape York (Australia), the Vedda (Sri Lanka), and many other "traditional" communities.

According to E. B. Tylor (1871), the Botocudos had only two real terms for numbers: one for "one", and the other for "a pair". With these lexical items they could manage to express three and four by saying something like "one and two" and "two and two". But these people had as much difficulty conceptualising a number above four as it is for us to imagine quantities of a trillion billions. For larger numbers, some of the Botocudos just pointed to their hair, as if to say "there are as many as there are hairs on my head".

A. Sommerfelt (1938) similarly reports that the Aranda had only two number-terms, *ninta* (one), and *tara* (two). Three and four were expressed as *tara-mi-ninta* (one and two) and *tara-ma-tara* ("two and two"), and the number series of the Aranda stopped there. For larger quantities, imprecise terms resembling "a lot", "several" and so on were used.

Likewise G. Hunt (1899) records the Murray islanders' use of the terms *netat* and *neis* for "one" and "two", and the expressions *neis-netat* (two + one) for "three", and *neis-neis* (two + two) for "four". Higher numbers were expressed by words like "a crowd of . . ."

Our final example is that of the Torres Straits islanders for whom *urapun* meant "one", *okosa* "two", *okosa-urapun* (two-one) "three", and *okosa-okosa* (two-two) "four". According to A. C. Haddon (1890) these were the only terms used for absolute quantities; other numbers were expressed by the word *ras*, meaning "a lot".

Attempts to teach such communities to count and to do arithmetic in the Western manner have frequently failed. There are numerous accounts of natives' lack of memory, concentration and seriousness when confronted with numbers and sums [see, for example, M. Dobrizhoffer (1902)]. It generally turned out much easier to teach primitive peoples the arts of music, painting, and sculpture than to get them to accept the interest and importance of arithmetic. This was perhaps not just because primitive peoples felt no need of counting, but also because numbers are amongst the most abstract concepts that humanity has yet devised. Children take longer to learn to do sums than to speak or to write. In the history of humanity, too, numbers have proved to be the hardest of these three skills.

PARITY BEFORE NUMBER

These primitive peoples nonetheless possessed a fundamental arithmetical rule which if systematically applied would have allowed them to manipulate numbers far in excess of four. The rule is what we call the *principle of base 2* (or binary principle). In this kind of numbering, five is "two-two-one", six is "two-two-two", seven is "two-two-two-one", and so on. But primitive societies did not develop binary numbering because, as L. Gerschel (1960) reminds us, they possessed only the most basic degree of numeracy, that which distinguishes between the singular and the dual.

A. C. Haddon (1890), observing the western Torres Straits islanders, noted that they had a pronounced tendency to count things in groups of two or in couples. M. Codrington, in *Melanesian Languages*, noticed the same thing in many Oceanic populations: "The natives of Duke of York's Island count in couples, and give the pairings different names depending how many of them there are; whereas in Polynesia, numbers are used although it is understood that they refer to so many pairs of things, not to so many things." Curr, as quoted by T. Dantzig (1930), confirms that Australian aborigines also counted in this way, to the extent that "if two pins are removed from a set of seven the aborigines rarely notice it, but they see straight away if only one is removed".

These primitive peoples obviously had a stronger sense of parity than of number. To express the numbers three and four, numbers they did not grasp as abstracts but which common sense allowed them to see in a single glance, they had recourse only to concepts of *one* and *pair*. And so for them groups like "two-one" or "two-two" were themselves pairs, not (as for us) the abstract integers (or "whole numbers") "three" and "four". So it is easy to see why they never developed the binary system to get as far as five and six, since these would have required three digits, one more than the pair which was their concept of the highest abstract number.

THE LIMITS OF PERCEPTION

The limited arithmetic of "primitive" societies does not mean that their members were unintelligent, nor that their innate abilities were or are lesser than ours. It would be a grave error to think that we could do better than a Torres Straits islander at recognising number if all we had to use were our natural faculties of perception.

In practice, when we want to distinguish a quantity we have recourse to our memories and/or to acquired techniques such as comparison, splitting, mental grouping, or, best of all, actual counting. For that reason it is rather difficult to get to our natural sense of number. There is an exercise that we can try, all the same. Looking at Fig. 1.1, which contains sets of objects *in line*, try to estimate the quantity of each set of objects in a single visual glance (that is to say, *without* counting). What is the best that we can do?

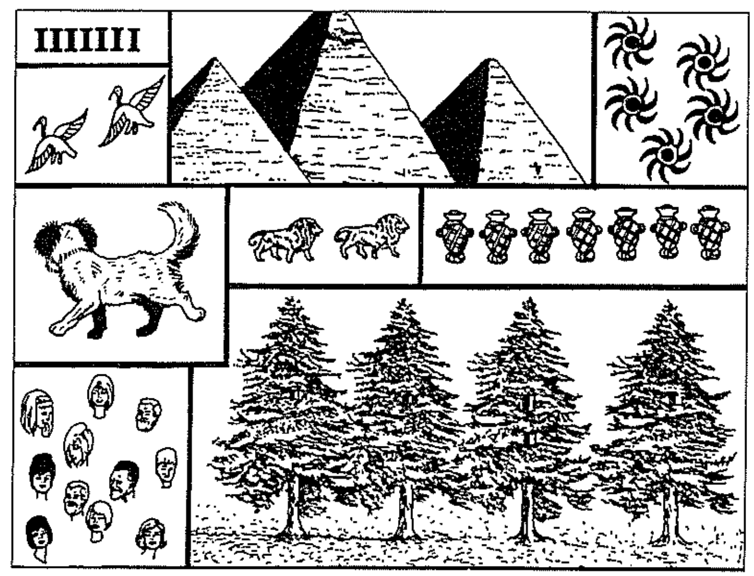


FIG. 1.1.

Everyone can see the sets of one, of two, and of three objects in the figure, and most people can see the set of four. But that's about the limit of our natural ability to numerate. Beyond four, quantities are vague, and our eyes alone cannot tell us how many things there are. Are there fifteen or twenty plates in that pile? Thirteen or fourteen cars parked along the street? Eleven or twelve bushes in that garden, ten or fifteen steps on this staircase, nine, eight or six windows in the façade of that house? The correct answers cannot be just seen. *We have to count to find out!*

The eye is simply not a sufficiently precise measuring tool: its natural number-ability virtually never exceeds four.

There are many traces of the "limit of four" in different languages and cultures. There are several Oceanic languages, for example, which distinguish between nouns in the singular, the dual, the triple, the quadruple, and the plural (as if in English we were to say *one bird, two birdo, three birdi, four birdu, many birds*).

In Latin, the names of the first four numbers (*unus, duos, tres, quatuor*) decline at least in part like other nouns and adjectives, but from five (*quinque*), Latin numerical terms are invariable. Similarly, Romans gave "ordinary" names to the first four of their sons (names like Marcus, Servius, Appius, etc.), but the fifth and subsequent sons were named only by a numeral: Quintus (the fifth), Sixtus (the sixth), Septimus (the seventh), and so on. In the original Roman calendar (the so-called "calendar of Romulus"), only the first four months had names (Martius, Aprilis, Maius, Junius), the fifth to tenth being referred to by their order-number: Quintilis, Sextilis, September, October, November, December.*

Perhaps the most obvious confirmation of the basic psychological rule of the "limit of four" can be found in the almost universal counting-device called (in England) the "five-barred gate". It is used by innkeepers keeping a tally or "slate" of drinks ordered, by card-players totting up scores, by prisoners keeping count of their days in jail, even by examiners working out the mark-distribution of a cohort of students:

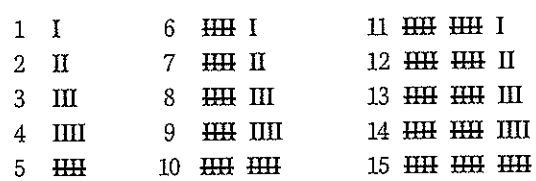


FIG. 1.2. *The five-barred gate*

* The original ten-month Roman calendar had 304 days and began with *Martius*. It was subsequently lengthened by the addition of two further months, *Januarius* and *Februarius* (our January and February). Julius Caesar further reformed the calendar, taking the start of the year back to 1 January and giving it 365 days in all. Later, the month of *Quintilis* was renamed *Julius* (our July) in honour of Caesar, and *Sextilis* became *Augustus* in honour of the emperor of that name.

Most human societies the world has known have used this kind of number-notation at some stage in their development and all have tried to find ways of coping with the unavoidable fact that beyond four (IIII) nobody can "read" intuitively a sequence of five strokes (IIIII) or more.

ARAMAIC (Egypt)
Elephantine script: 5th to 3rd centuries BCE

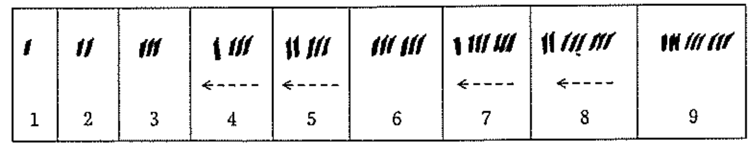


FIG. 1.3.

ARAMAIC (Mesopotamia)
Khatra script: First decades of CE

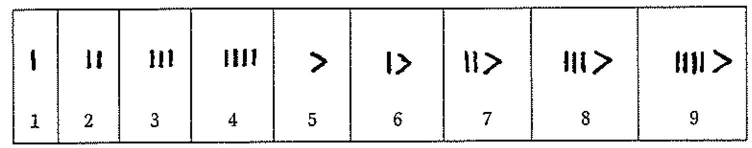


FIG. 1.4.

ARAMAIC (Syria)
Palmyrenean script: First decades of CE

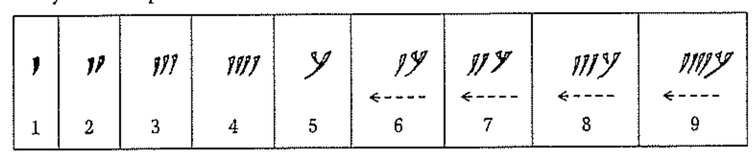


FIG. 1.5.

CRETAN CIVILISATION
Hieroglyphic script: first half of second millennium BCE

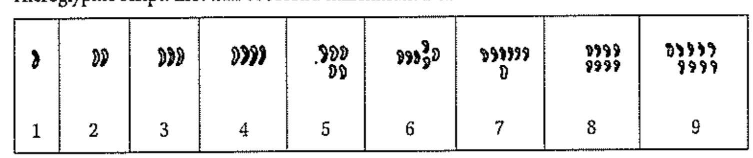


FIG. 1.6.

CRETAN CIVILISATION
Linear script: 1700–1200 BCE

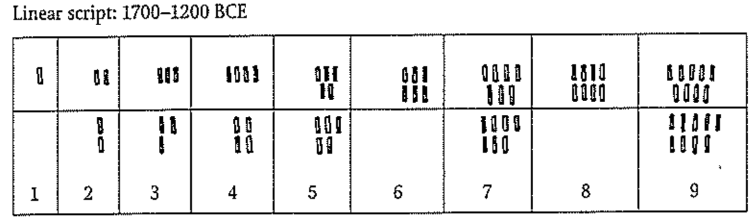


FIG. 1.7.

EGYPT
Hieroglyphic script: third to first millennium BCE

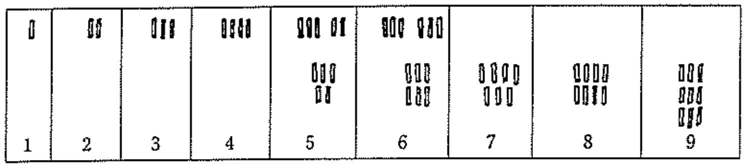


FIG. 1.8.

ELAM
"Proto-Elamite" script: Iran, first half of third millennium BCE

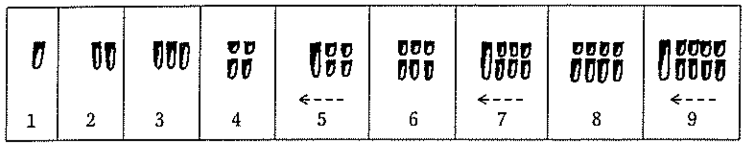


FIG. 1.9.

ETRUSCAN CIVILISATION
Italy, 6th to 4th centuries BCE

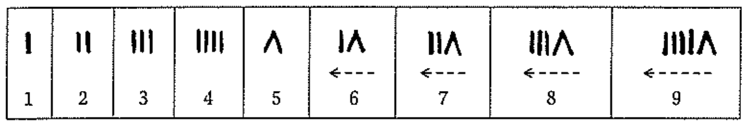


FIG. 1.10.

GREECE
Epidaurus and Argos, 5th to 2nd centuries BCE

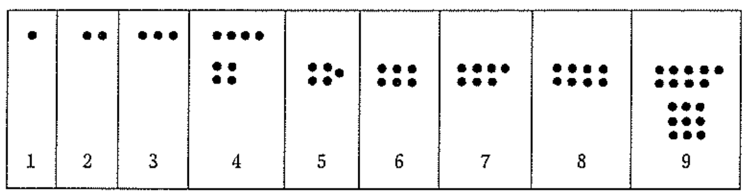
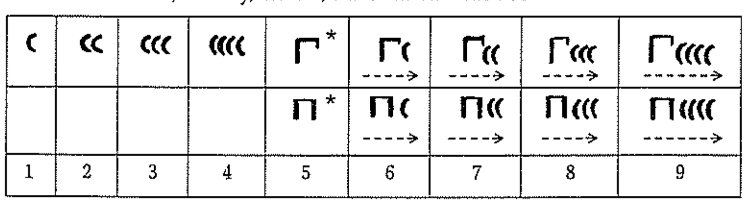


FIG. 1.11.

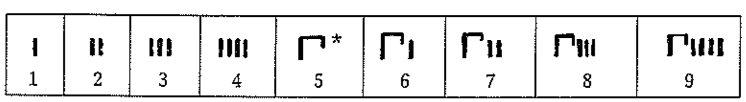
GREECE
Taurian Chersonesus, Chalcidy, Troezen, 5th to 2nd centuries BCE



*π, initial of pente, five

FIG. 1.12.

GREECE
Thebes, Karistos, 5th to 1st centuries BCE



*π, initial of pente, five

FIG. 1.13.

INDUS CIVILISATION
2300–1750 BCE

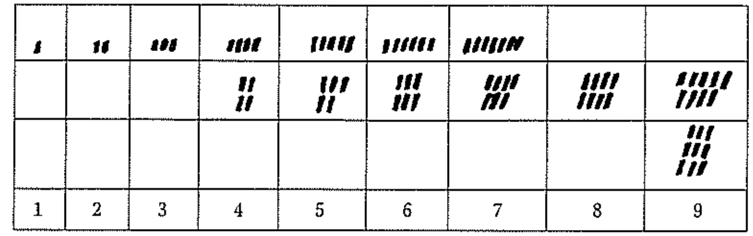


FIG. 1.14.

HITTITE CIVILISATION
Hieroglyphic: Anatolia, 1500–800 BCE

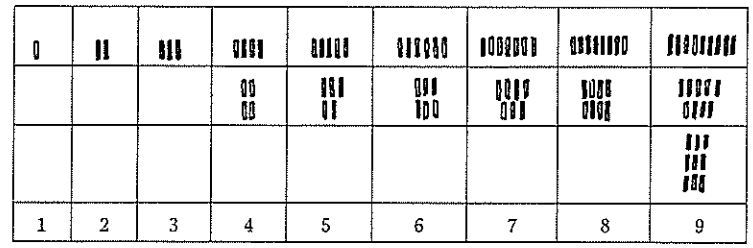


FIG. 1.15.

LYCIAN CIVILISATION
Asia Minor, first half of first millennium BCE

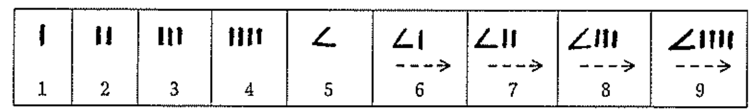


FIG. 1.16.

LYDIAN CIVILISATION
Asia Minor, 6th to 4th centuries BCE

FIG. 1.17.

MAYAN CIVILISATION
Pre-Columbian Central America, 3rd to 14th centuries CE

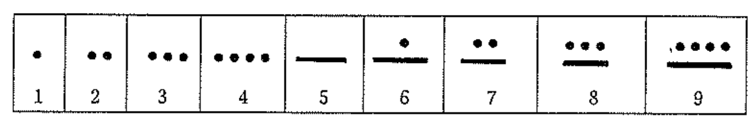


FIG. 1.18.

MESOPOTAMIA
Archaic Sumerian, beginning of third millennium BCE

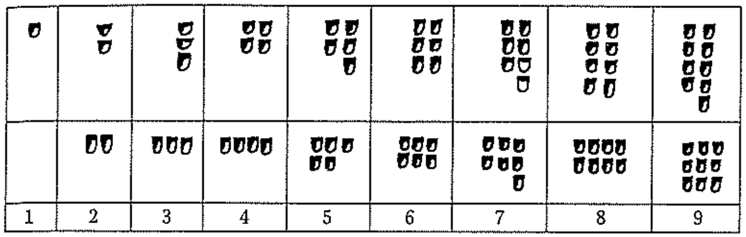


FIG. 1.19.

MESOPOTAMIA
Sumerian cuneiform, 2850–2000 BCE

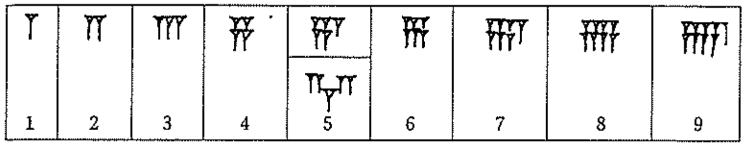


FIG. 1.20.

MESOPOTAMIA
Assyro-Babylonian cuneiform, second to first millennium BCE

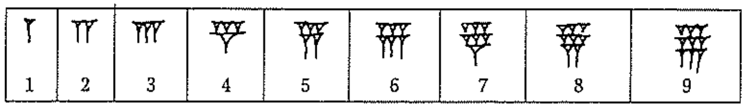


FIG. 1.21.

CIVILISATIONS OF MA'IN & SABA (SHEBA)
Southern Arabia, 5th to 1st centuries BCE

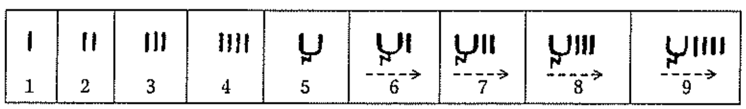


FIG. 1.22.

PHOENICIAN CIVILISATION
From 6th century BCE

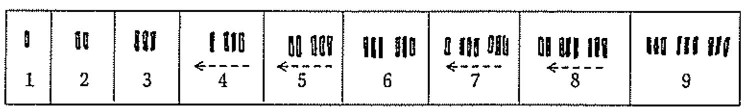


FIG. 1.23.

URARTU
Hieroglyphic script, Armenia, 13th to 9th centuries BCE

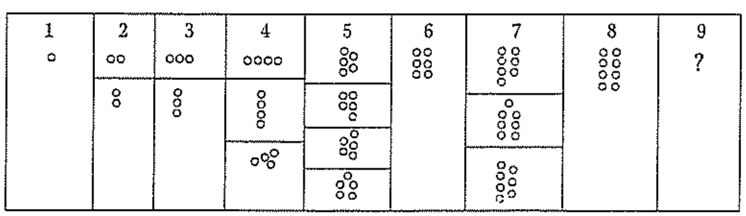


FIG. 1.24.

To recapitulate: at the start of this story, people began by counting the first nine numbers by placing in sequence the corresponding number of strokes, circles, dots or other similar signs representing "one", more or less as follows:

I	II	III	IIII	IIIII	IIIIII	IIIIIII	IIIIIIII	IIIIIIIII
1	2	3	4	5	6	7	8	9

FIG. 1.25.

But because series of identical signs are not easy to read quickly for numbers above four, the system was rapidly abandoned. Some civilisations (such as those found in Egypt, Sumer, Elam, Crete, Urartu, and Greece) got round the difficulty by grouping the signs for numbers from five to nine to 9 according to a principle that we might call *dyadic representation:*

I	II	III	IIII	III	III	IIII	IIII	IIIII
				II	III	III	IIII	IIII
1	2	3	4	5	6	7	8	9
				(3 + 2)	(3 + 3)	(4 + 3)	(4 + 4)	(5 + 4)

FIG. 1.26.

Other civilisations, such as the Assyro-Babylonian, the Phoenician, the Egyptian-Aramaean and the Lydian, solved the problem by recourse to a *rule of three:*

I	II	III	III	III	III	III	III	III
			I	II	III	III	III	III
						I	II	III
1	2	3	4	5	6	7	8	9
			(3 + 1)	(3 + 2)	(3 + 3)	(3 + 3 + 1)	(3 + 3 + 2)	(3 + 3 + 3)

FIG. 1.27.

And yet others, like the Greeks, the Manaeans and Sabaeans, the Lycians, Mayans, Etruscans and Romans, came up with an idea (probably based on finger-counting) for a special sign for the number five, proceeding thereafter on a *rule of five* or quinary system (6 = 5 + 1, 7 = 5 + 2, and so on).

There really can be no debate about it now: *natural human ability to perceive number does not exceed four!*

So the basic root of arithmetic as we know it today is a very rudimentary numerical capacity indeed, a capacity barely greater than that of some animals. There's no doubt that the human mind could no more accede *by innate aptitude alone* to the abstraction of counting than could crows or goldfinches. But human societies have enlarged the potential of these very limited abilities by inventing a number of mental procedures of enormous fertility, procedures which opened up a pathway into the universe of numbers and mathematics . . .

DEAD RECKONING

Since we can discriminate unreflectingly between concrete quantities only up to four, we cannot have recourse only to our natural sense of number to get to any quantity greater than four. We must perforce bring into play the device of abstract counting, the characteristic quality of "civilised" humanity.

But is it therefore the case that, in the absence of this mental device for counting (in the way we now understand the term), the human mind is so enfeebled that it cannot engage in any kind of numeration at all?

It is certainly true that without the abstractions that we call "one", "two", "three", and so on it is not easy to carry out mental operations. But it does not follow at all that a mind without numbers of our kind is incapable of devising specific tools for manipulating quantities in concrete sets. There are very good reasons for thinking that for many centuries people were able to reach several numbers without possessing anything like number-concepts.

There are many ethnographic records and reports from various parts of Africa, Oceania and the Americas showing that numerous contemporary "primitive" populations have numerical techniques that allow them to carry out some "operations", at least to some extent.

These techniques, which, in comparison to our own, could be called "concrete", enable such peoples to reach the same results as we would, by using *mediating objects* or *model collections* of many different kinds (pebbles, shells, bones, hard fruit, dried animal dung, sticks, the use of notched bones or sticks, etc.). The techniques are much less powerful and often more complicated than our own, but they are perfectly serviceable for establishing (for example) whether as many head of cattle have returned from grazing as went out of the cowshed. You do not need to be able to count by numbers to get the right answer for problems of that kind.

ELEMENTARY ARITHMETIC

It all started with the device known as "one-for-one correspondence". This allows even the simplest of minds to compare two collections of beings or things, of the same kind or not, without calling on an ability to count in numbers. It is a device which is both the prehistory of arithmetic, and the dominant mode of operation in all contemporary "hard" sciences.

Here is how it works: You get on a bus and you have before you (apart from the driver, who is in a privileged position) two sets: a set of seats and a set of passengers. In one glance you can tell whether the two sets have "the same number" of elements; and, if the two sets are not equal, you can tell just as quickly which is the larger of the two. This ready-reckoning of number without recourse to numeration is more easily explained by the device of one-for-one correspondence.

If there was no one standing in the bus and there were some empty seats, you would know that each passenger has a seat, but that each seat does not necessarily have a passenger: therefore, there are fewer passengers than seats. In the contrary case – if there are people standing and all the seats are taken – you know that there are more passengers than seats. The third possibility is that there is no one standing and all seats are taken: as each seat corresponds to one passenger, there are as many passengers as seats. The last situation can be described by saying that there is a mapping (or a *biunivocal correspondence*, or, in terms of modern mathematics, a *bijection*) between the number of seats and the number of passengers in the bus.

At about fifteen or sixteen months, infants go beyond the stage of simple observation of their environment and become capable of grasping the principle of one-for-one correspondence, and in particular the property of mapping. If we give a baby of this age equal numbers of dolls and little chairs, the infant will probably try to fit one doll on each seat. This kind of play is nothing other than mapping the elements of one set (dolls) onto the elements of a second set (chairs). But if we set out more dolls than chairs (or more chairs than dolls), after a time the baby will begin to fret: it will have realised that the mapping isn't working.

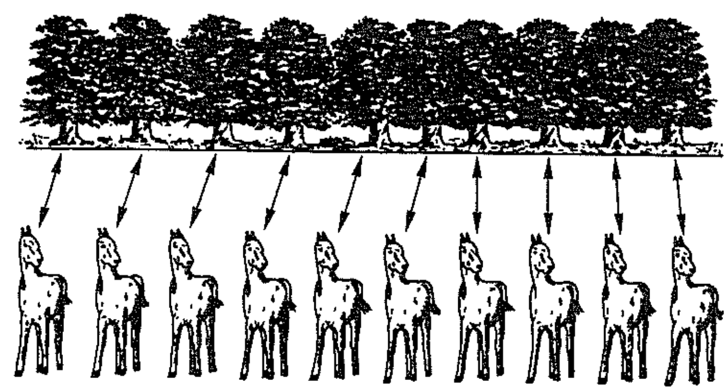


FIG. 1.28. *Two sets map if for each element of one set there is a corresponding single element of the other, and vice versa.*

This mental device does not only provide a means for comparing two groups, but it also allows its user to manipulate several numbers without knowing how to count or even to name the quantities involved.

If you work at a cinema box-office you usually have a seating plan of the auditorium in front of you. There is one "box" on the plan for each seat in the auditorium, and, each time you sell a ticket, you cross out one of the boxes on the plan. What you are doing is: mapping the seats in the cinema onto the boxes on the seating plan, then mapping the boxes on the plan onto the tickets sold, and finally, mapping the tickets sold onto the number of people allowed into the auditorium. So even if you are too lazy to add up the number of tickets you've sold, you'll not be in any doubt about knowing when the show has sold out.

To recite the attributes of Allah or the obligatory laudations after prayers, Muslims habitually use a string of prayer-beads, each bead corresponding to one divine attribute or to one laudation. The faithful "tell their beads" by slipping a bead at a time through their fingers as they proceed through the recitation of eulogies or of the attributes of Allah.

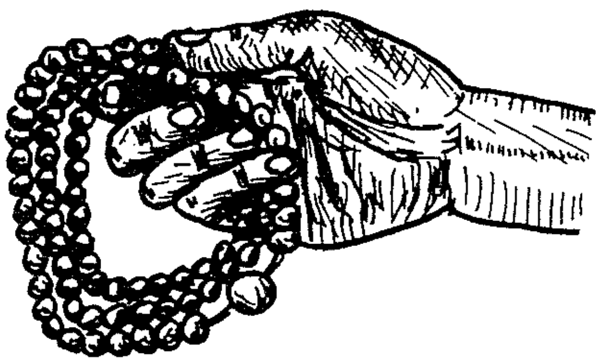


FIG. 1.29. *Muslim prayer-beads* (subha *or* sebha *in Arabic) used for reciting the 99 attributes of Allah or for supererogatory laudations. This indispensable piece of equipment for pilgrims and dervishes is made of wooden, mother-of-pearl or ivory beads that can be slipped through the fingers. It is often made up of three groups of beads, separated by two larger "marker" beads, with an even larger bead indicating the start. There are usually a hundred beads on a string (33 + 33 + 33 + 1), but the number varies.*

Buddhists have also used prayer-beads for a very long time, as have Catholics, for reciting *Pater noster*, *Ave Maria*, *Gloria Patri*, etc. As these litanies must be recited several times in a quite precise order and number, Christian rosaries usually consist of a necklace threaded with five times ten small beads, each group separated by a slightly larger bead, together with a chain bearing one large then three small beads, then one large bead and a cross. That is how the litanies can be recited without counting but without omission – each small bead on the ring corresponds to one *Ave Maria*, with a *Gloria Patri* added on the last bead of each set of ten, and a *Pater noster* is said for each large bead, and so on.

The device of one-for-one correspondence has thus allowed these religions to devise a system which ensures that the faithful do not lose count of their litanies despite the considerable amount of repetition required. The device can thus be of use to the most "civilised" of societies; and for the completely "uncivilised" it is even more valuable.

Let us take someone with no arithmetical knowledge at all and send him to the grocery store to get ten loaves of bread, five bottles of cooking oil, and four bags of potatoes. With no ability to count, how could this person be trusted to bring back the correct amount of change? But in fact such a person is perfectly capable of carrying out the errand provided the proper equipment is available. The appropriate kit is necessarily based on the principle of one-for-one correspondence. We could make ten purses out of white cloth, corresponding to the ten loaves, five yellow purses for the bottles of cooking oil, and four brown purses, for the bags of potatoes. In each purse we could put the exact price of the corresponding item of purchase, and all the uneducated shopper needs to know is that a white purse can be exchanged for a loaf, a yellow one for a bottle of oil and a brown one for a bag of potatoes.

This is probably how prehistoric humanity did arithmetic for many millennia, before the first glimmer of arithmetic or of number-concepts arose.

Imagine a shepherd in charge of a flock of sheep which is brought back to shelter every night in a cave. There are fifty-five sheep in this flock. But the shepherd doesn't know that he has fifty-five of them since he does not know the number "55": all he knows is that he has "many sheep". Even so, he wants to be sure that all his sheep are back in the cave each night. So he has an idea – the idea of a concrete device which prehistoric humanity used for many millennia. He sits at the mouth of his cave and lets the animals in one by one. He takes a flint and an old bone, and cuts a notch in the bone for every sheep that goes in. So, without realising the mathematical meaning of it, he has made exactly fifty-five incisions on the bone by the time the last animal is inside the cave. Henceforth the shepherd can check whether any sheep in his flock are missing. Every time he comes back from grazing, he lets the sheep into the cave one by one, and moves his finger over one indentation in the tally stick for each one. If there are any marks left on the bone after the last sheep is in the cave, that means he has lost some sheep. If not, all is in order. And if meanwhile a new lamb comes along, all he has to do is to make another notch in the tally bone.

So thanks to the principle of one-for-one correspondence it is possible to manage to count even in the absence of adequate words, memory or abstraction.

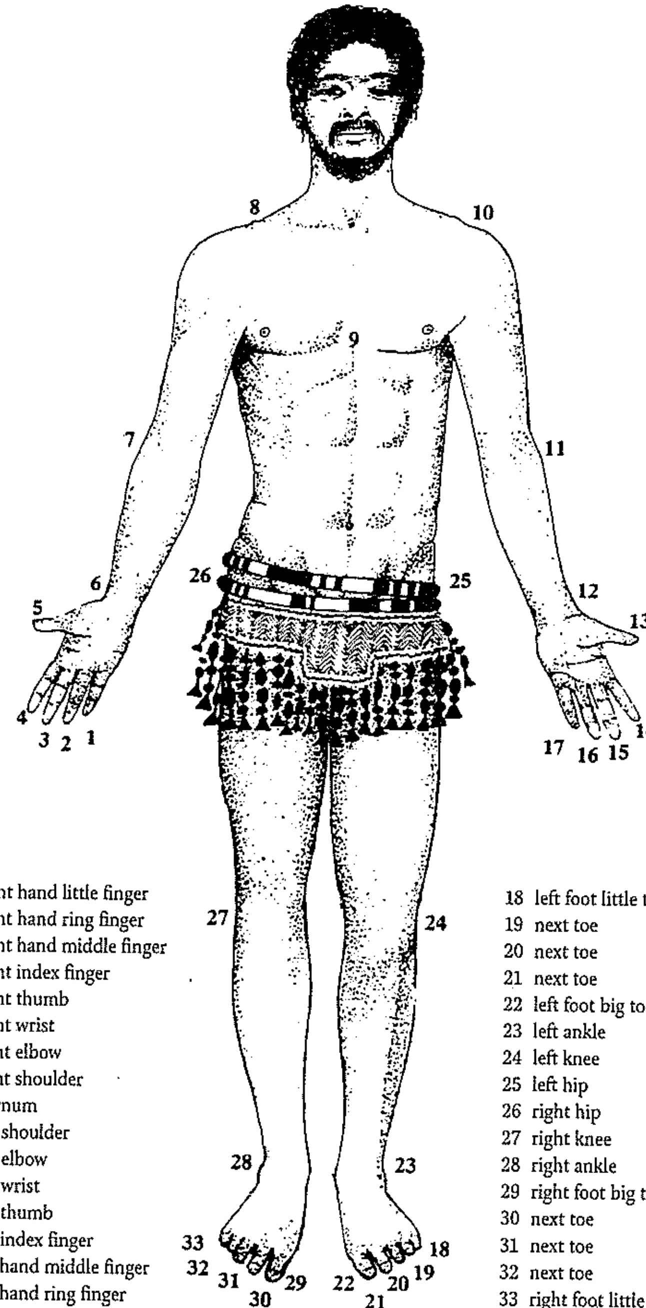
One-for-one mapping of the elements of one set onto the elements of a second set creates an abstract idea, entirely independent of the type or nature of the things or beings in the one or other set, which expresses a property common to the two sets. In other words, mapping abolishes the distinction that holds between two sets by virtue of the type or nature of the elements that constitute them. This abstract property is precisely why one-for-one mapping is a significant tool for tasks involving enumeration; but in practice, the methods that can be based on it are only suitable for relatively small sets.

This is why *model collections* can be very useful in this domain. Tally sticks with different numbers of marks on them constitute so to speak a range of *ready-made mappings* which can be referred to independently of the type or nature of the elements that they originally referred to. A stick of ivory or wood with twenty notches on it can be used to enumerate twenty men, twenty sheep or twenty goats just as easily as it can be used for twenty bison, twenty horses, twenty days, twenty pelts, twenty kayaks, or twenty measures of grain. The only number technique that can be built on this consists of choosing the most appropriate tally stick from the ready-mades so as to obtain a one-to-one mapping on the set that you next want to count.

However, notched sticks are not the only concrete *model collections* available for this kind of matching-and-counting. The shepherd of our example could also have used pebbles for checking that the same number of sheep come into the cave every evening as went out each morning. All he needs to do to use this device would be to associate one pebble with each head of sheep, to put the resulting pile of pebbles in a safe place, and then to count them out in a reverse procedure on returning from the pasture. If the last animal in matches the last pebble in the pile, then the shepherd knows for sure that none of his flock has been lost, and if a lamb has been born meanwhile, all he needs to do is to add a pebble to the pile.

All over the globe people have used a variety of objects for this purpose: shells, pearls, hard fruit, knucklebones, sticks, elephant teeth, coconuts, clay pellets, cocoa beans, even dried dung, organised into heaps or lines corresponding in number to the tally of the things needing to be checked. Marks made in sand, and beads and shells, strung on necklaces or made into rosaries, have also been used for keeping tallies.

Even today, several "primitive" communities use parts of the body for this purpose. Fingers, toes, the articulations of the arms and legs (elbow, wrist, knee, ankle . . .), eyes, nose, mouth, ears, breasts, chest, sternum, hips and so on are used as the reference elements of one-for-one counting systems. Much of the evidence comes from the Cambridge Anthropological Expedition to Oceania at the end of the last century. According to Wyatt Gill, some Torres Straits islanders "counted visually" (see Fig. 1.30):



1 right hand little finger
2 right hand ring finger
3 right hand middle finger
4 right index finger
5 right thumb
6 right wrist
7 right elbow
8 right shoulder
9 sternum
10 left shoulder
11 left elbow
12 left wrist
13 left thumb
14 left index finger
15 left hand middle finger
16 left hand ring finger
17 left hand little finger
18 left foot little toe
19 next toe
20 next toe
21 next toe
22 left foot big toe
23 left ankle
24 left knee
25 left hip
26 right hip
27 right knee
28 right ankle
29 right foot big toe
30 next toe
31 next toe
32 next toe
33 right foot little toe

FIG. 1.30. *Body-counting system used by Torres Straits islanders*

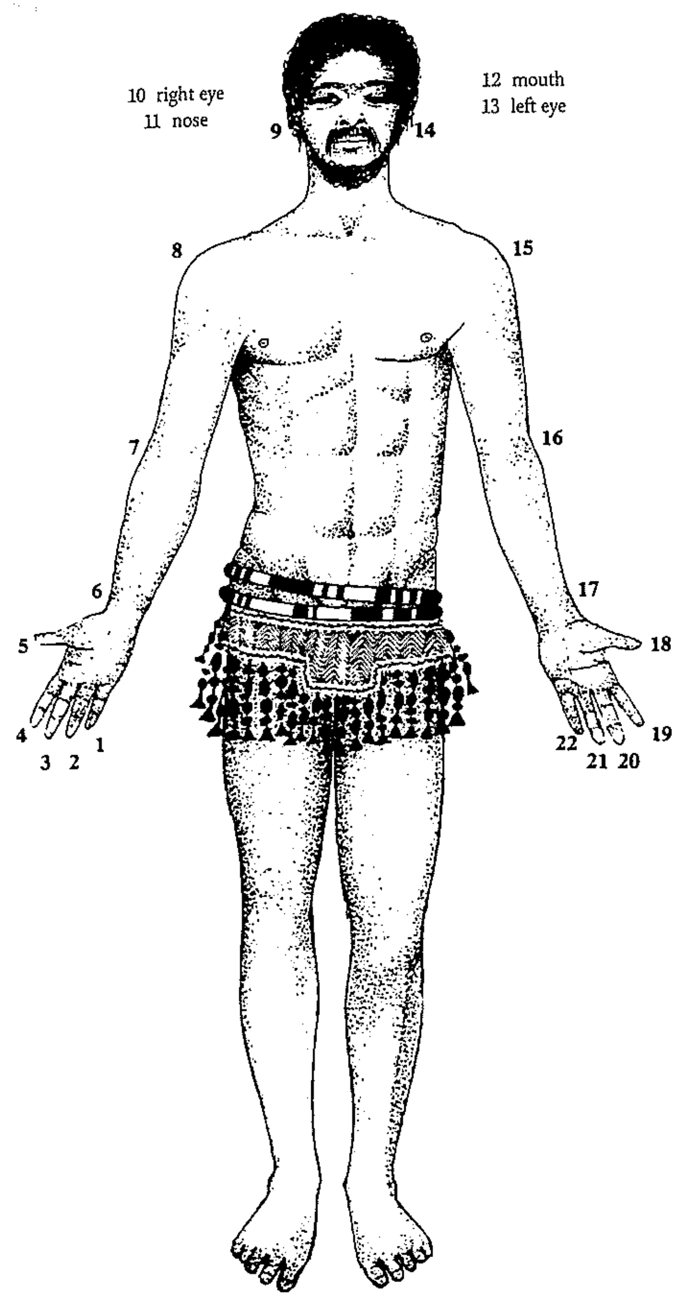


FIG. 1.31. *System used by Papuans (New Guinea)*

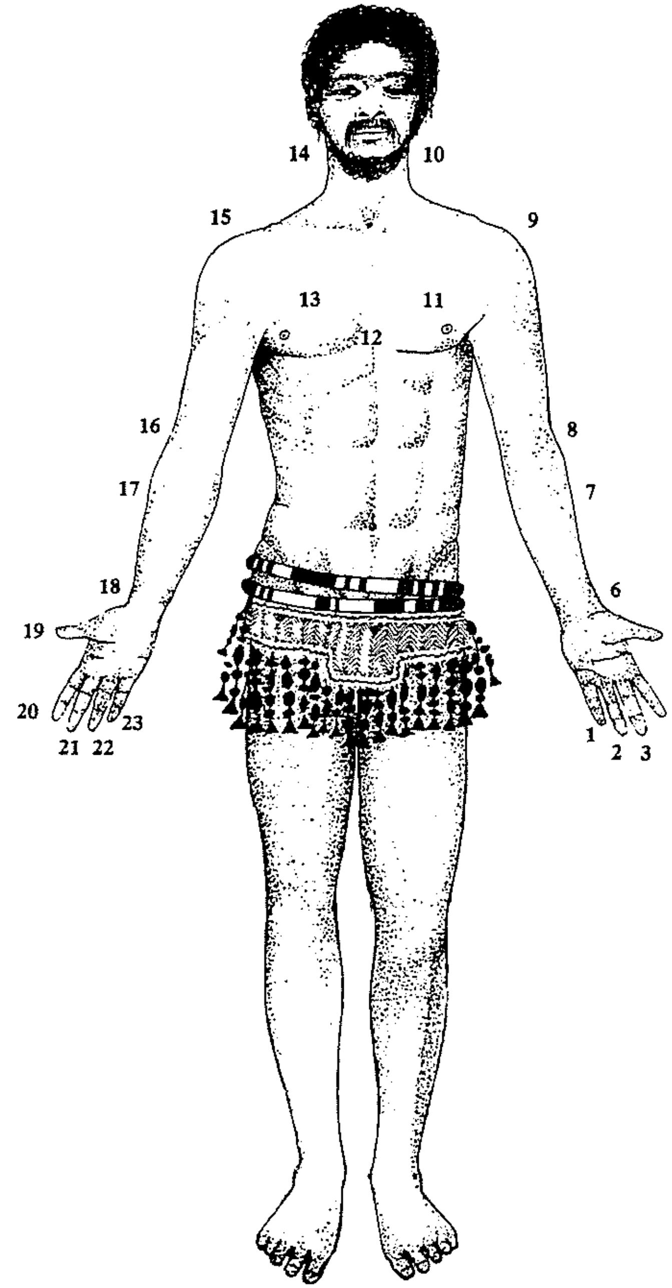


FIG. 1.32. *Body-counting system used by the Elema (New Guinea)*

They touch first the fingers of their right hand, one by one, then the right wrist, elbow and shoulder, go on to the sternum, then the left-side articulations, not forgetting the fingers. This brings them to the number seventeen. If the total needed is higher, they add the toes, ankle, knee and hip of the left then the right hand side. That gives 16 more, making 33 in all. For even higher numbers, the islanders have recourse to a bundle of small sticks. [As quoted in A. C. Haddon (1890)]

Murray islanders also used parts of the body in a conventional order, and were able to reach 29 in this manner. Other Torres Straits islanders used similar procedures which enabled them to "count visually" up to 19; the same customs are found amongst the Papuans and Elema of New Guinea.

NUMBERS, GESTURES, AND WORDS

The question arises: is the mere enumeration of parts of the body in regular order tantamount to a true arithmetical sequence? Let us try to find the answer in some of the ethnographic literature relating to Oceania.

The first example is from the Papuan language spoken in what was British New Guinea. According to the report of the Cambridge Expedition to the Torres Straits, Sir William MacGregor found that "body-counting" was prevalent in all the villages below the Musa river. "Starting with the little finger on the right hand, the series proceeds with the right-hand fingers, then the right wrist, elbow, shoulder, ear and eye, then on to the left eye, and so on, down to the little toe on the left foot." Each of the gestures to these parts of the body is accompanied, the report continues, by a specific term in Papuan, as follows:

NUMBER	NUMBER-GESTURE	GESTURE-WORD
1	right hand little finger	*anusi*
2	right hand ring finger	*doro*
3	right hand middle finger	*doro*
4	right hand index finger	*doro*
5	right thumb	*ubei*
6	right wrist	*tama*
7	right elbow	*unubo*
8	right shoulder	*visa*
9	right ear	*denoro*
10	right eye	*diti*
11	left eye	*diti*
12	nose	*medo*
13	mouth	*bee*
14	left ear	*denoro*
15	left shoulder	*visa*
16	left elbow	*unubo*
17	left wrist	*tama*
18	left thumb	*ubei*
19	left hand index finger	*doro*
20	left hand middle finger	*doro*
21	left hand ring finger	*doro*
22	left hand little finger	*anusi*

The words used are simply the names of the parts of the body, and strictly speaking they are not numerical terms at all. *Anusi*, for example, is associated with both 1 and 22, and is used to indicate the little fingers of both the right and the left hands. In these circumstances how can you know which number is meant? Similarly the term *doro* refers to the ring, middle and index fingers of both hands and "means" either 2 or 3 or 4 or 19 or 20 or 21. Without the accompanying gesture, how could you possibly tell which of these numbers was meant?

However, there is no ambiguity in the system. What is spoken is the name of the part of the body, which has its rank-order in a fixed, conventional sequence within which no confusion is possible. So there is no doubt that the mere enumeration of the parts of the body does not constitute a true arithmetical sequence unless it is associated with a corresponding sequence of gestures. Moreover, the mental counting process has no direct oral expression – you can get to the number required without uttering a word. A conventional set of "number-gestures" is all that is needed.

In those cases where it is possible to recover the original meanings of the names given to numbers, it often turns out that they retain traces of body-counting systems like those we have looked at. Here, for example, are the number-words used by the Bugilai (former British New Guinea) together with their etymological meanings:

1	*tarangesa*	left hand little finger
2	*meta kina*	next finger
3	*guigimeta kina*	middle finger
4	*topea*	index finger
5	*manda*	thumb
6	*gaben*	wrist
7	*trankgimbe*	elbow
8	*podei*	shoulder
9	*ngama*	left breast
10	*dala*	right breast

[Source: J. Chalmers (1898)]

E. C. Hawtrey (1902) also reports that the Lengua people of the Chaco (Paraguay) use a set of number-names broadly derived from specific number-gestures. Special words apparently unrelated to body-counting are used for 1 and 2, but for the other numbers they say something like:

3	"made of one and two"
4	"both sides same"
5	"one hand"
6	"reached other hand, one"
7	"reached other hand, two"
8	"reached other hand, made of one and two"
9	"reached other hand, both sides same"
10	"finished, both hands"
11	"reached foot, one"
12	"reached foot, two"
13	"reached foot, made of one and two"
14	"reached foot, both sides same"
15	"finished, foot"
16	"reached other foot, one"
17	"reached other foot, two"
18	"reached other foot, made of one and two"
19	"reached other foot, both sides same"
20	"finished, feet"

The Zuñi have names for numbers which F. H. Cushing (1892) calls "manual concepts":

1	*töpinte*	taken to begin
2	*kwilli*	raised with the previous
3	*kha'i*	the finger that divides equally
4	*awite*	all fingers raised bar one
5	*öpte*	the scored one
6	*topalïk'ye*	another added to what is counted already
7	*kwillik'ya*	two brought together and raised with the others
8	*khailïk'ya*	three brought together and raised with the others
9	*tenalïk'ya*	all bar one raised with the others
10	*ästem'thila*	all the fingers
11	*ästem'thila topayä'thl' tona*	all the fingers and one more raised

and so on.

All this leads us to suppose that in the remotest past gestures came before any oral expression of numbers.

CARDINAL RECKONING DEVICES FOR CONCRETE QUANTITIES

Let us now imagine a group of "primitive" people lacking any conception of abstract numbers but in possession of perfectly adequate devices for "reckoning" relatively small sets of concrete objects. They use all sorts of model collections, but most often they "reckon by eye" in the following manner: they touch each other's right-hand fingers, starting with the little finger, then the right wrist, elbow, shoulder, ear, and eye. Then they touch each others' nose, mouth, then the left eye, ear, shoulder, elbow, and wrist, and on to the little finger of the left hand, getting to 22 so far. If the number needed is higher, they go on to the breasts, hips, and genitals, then the knees, ankles and toes on the right then the left sides. This extension allows 19 further integers, or a total of 41.

The group has recently skirmished with a rebellious neighbouring village and won. The group's leader decides to demand reparations, and entrusts one of his men with the task of collecting the ransom. "For each of the warriors we have lost", says the chief, "they shall give us as many pearl necklaces as there are from the little finger on my right hand to my right eye, as many pelts as there are from the little finger of my right hand to my mouth, and as many baskets of food as there are from the little finger of my right hand to my left wrist." What this means is that the reparation for each lost soldier is:

10 pearl necklaces
12 pelts
17 baskets of food

In this particular skirmish, the group lost sixteen men. Of course none amongst the group has a notion of the number "16", but they have an infallible method of determining numbers in these situations: on departing for the fight, each warrior places a pebble on a pile, and on his return each surviving warrior picks a pebble out of the pile. The number of unclaimed pebbles corresponds precisely to the number of warriors lost.

One of the leader's envoys then takes possession of the pile of remaining pebbles but has them replaced by a matching bundle of sticks, which is easier to carry. The chief checks the emissaries' equipment and their comprehension of the reparations required, and sends them off to parley with the enemy.

The envoys tell the losing side how much they owe, and proceed to enumerate the booty in the following manner: one steps forward and says:

"Bring me a pearl necklace each time I point to a part of my body," and he then touches in order the little finger, the ring finger, the middle finger, the index finger and the thumb of his right hand. So the vanquished bring him one necklace, then a second, then a third and so on up to the fifth. The envoy then repeats himself, but pointing to his right wrist, elbow, shoulder, ear and eye, which gets him five more necklaces. So without having any concept of the number "10" he obtains precisely ten necklaces.

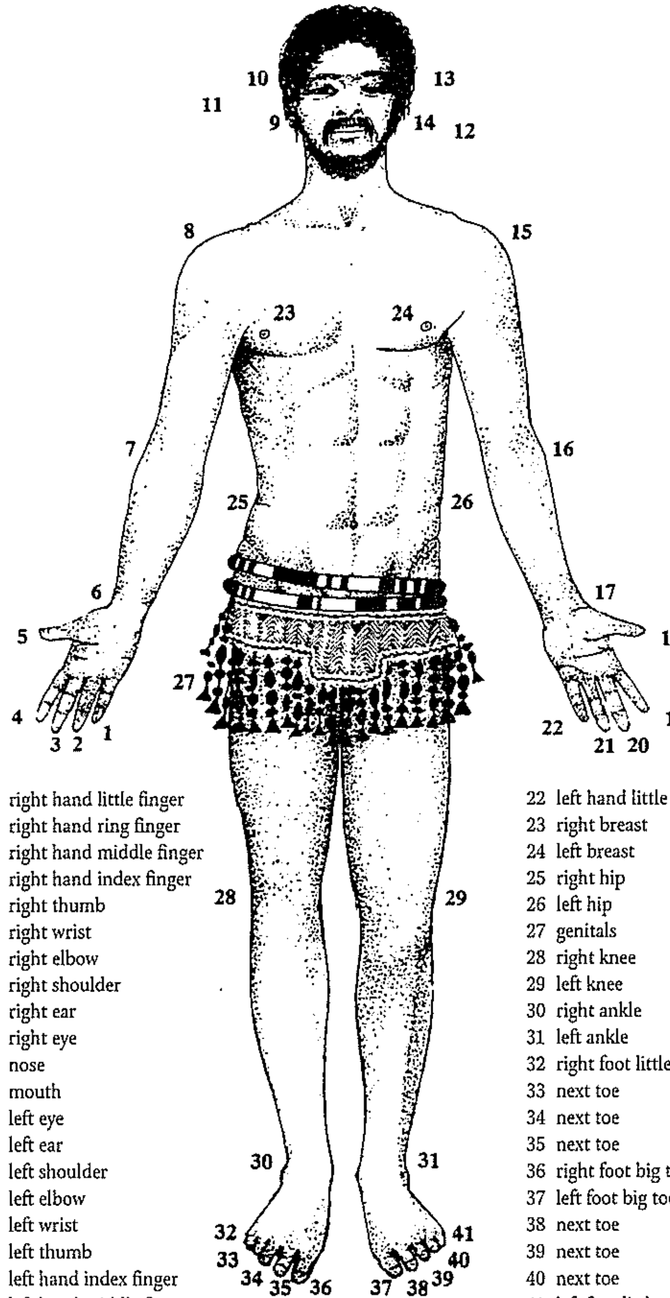
Another envoy proceeds in identical fashion to obtain the twelve pelts, and a third takes possession of the seventeen baskets of food that are demanded.

That is when the fourth envoy comes into the equation, for he possesses the tally of warriors lost in the battle, in the form of a bundle of sixteen sticks. He sets one aside, and the three other envoys then repeat their operations, allowing him to set another stick aside, and so on, until there are no sticks left in the bundle. That is how they know that they have the full tally, and so collect up the booty and set off with it to return to their own village.

As can be seen, "primitives" of this kind are not using body-counting in exactly the same way as we might. Since we know how to count, a conventional order of the parts of the body would constitute a true arithmetical sequence; each "body-point" would be assimilated in our minds to a cardinal (rank-order) number, characteristic of a particular quantity of things or beings. For instance, to indicate the length of a week using this system, we would not need to remember that it contained as many days as mapped onto our bodies from the right little finger to the right elbow, since we could just attach to it the "rank-order number" called "right elbow", which would suffice to symbolise the numerical value of any set of seven elements.

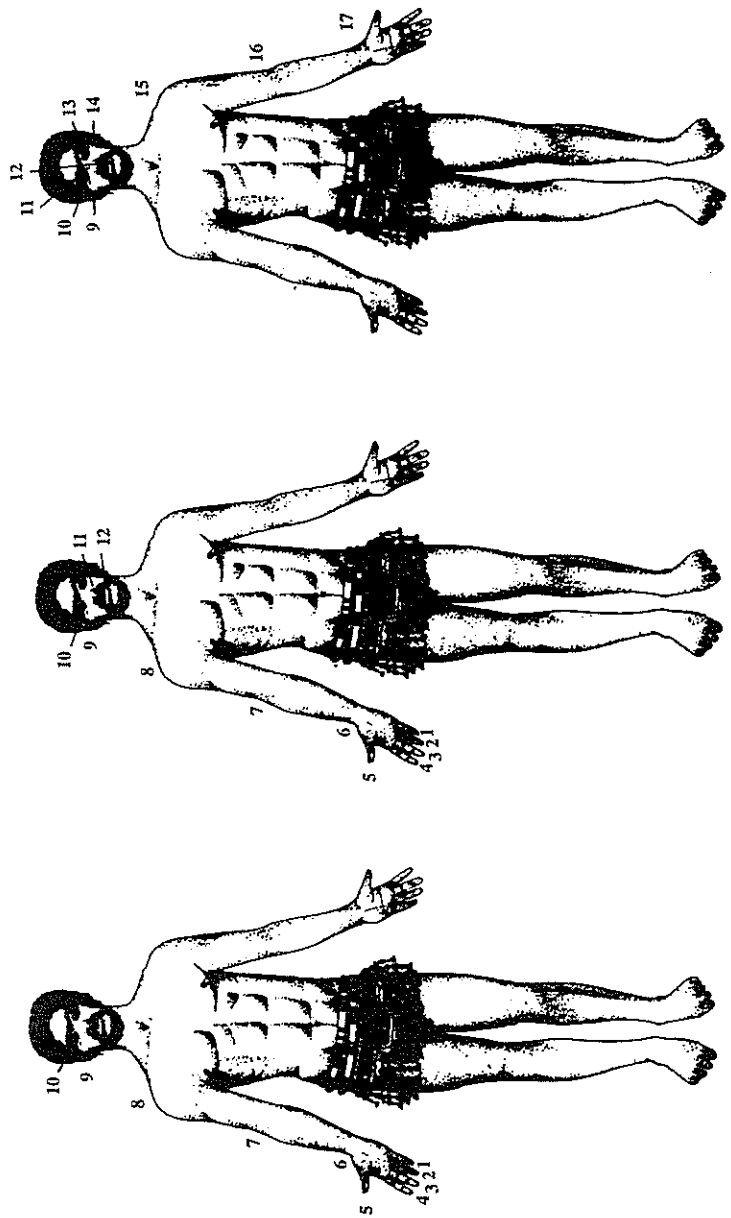
That is because we are equipped with *generalising abstractions* and in particular with number-concepts. But "primitive" peoples are not so equipped: they cannot *abstract* from the "points" in the numbering sequence: their grasp of the sequence remains embedded in the specific nature of the "points" themselves. Their understanding is in effect restricted to one-for-one mapping; the only "operations" they make are to add or remove one or more of the elements in the basic series.

Such people do not of course have any abstract concept of the number "ten", for instance. But they do know that by touching in order their little finger, ring finger, middle finger, index finger and thumb on the right hand, then their right wrist, elbow, shoulder, ear, and eye, they can "tally out" as many men, animals or objects as there are body-points in the sequence. And having done so, they remember perfectly well which body-point any particular tally of things or people reached, and are able to repeat the operation in order to reach exactly the same tally whenever they want to.



1 right hand little finger
2 right hand ring finger
3 right hand middle finger
4 right hand index finger
5 right thumb
6 right wrist
7 right elbow
8 right shoulder
9 right ear
10 right eye
11 nose
12 mouth
13 left eye
14 left ear
15 left shoulder
16 left elbow
17 left wrist
18 left thumb
19 left hand index finger
20 left hand middle finger
21 left hand ring finger
22 left hand little finger
23 right breast
24 left breast
25 right hip
26 left hip
27 genitals
28 right knee
29 left knee
30 right ankle
31 left ankle
32 right foot little toe
33 next toe
34 next toe
35 next toe
36 right foot big toe
37 left foot big toe
38 next toe
39 next toe
40 next toe
41 left foot little toe

FIG. 1.33.



Counting the ten necklaces

Counting the twelve pelts

Counting the seventeen baskets of food

FIG. 1.34.

In other words, this procedure is a simple and convenient means of establishing ready-made mappings which can then be mapped one-to-one onto any sets for which a total is required. So when our imaginary tribe went to collect its ransom, they used only these notions, not any true number-concepts. They simply mapped three such ready-made sets onto a set of ten necklaces, a set of twelve pelts, and a set of seventeen baskets of food for each of the lost warriors.

These body-counting points are thus not thought of by their users as "numbers", but rather as the last elements of model sets arrived at after a regulated (conventional) sequence of body-gestures. This means that for such people the mere designation of any one of the points *is not sufficient to describe a given number of beings or things unless the term uttered is accompanied by the corresponding sequence of gestures*. So in discussions concerning such and such a number, no real "number-term" is uttered: instead, a given number of body-counting points will be enumerated, alongside the simultaneous sequence of gestures. This kind of enumeration therefore fails to constitute a genuine arithmetical series; participants in the discussion must also necessarily keep their eyes on the speaker!

All the same, our imaginary tribesmen have unknowingly reached quite large numbers, even with such limited tools, since they have collected:

$16 \times 10 = 160$ necklaces
$16 \times 12 = 192$ pelts
$16 \times 17 = 272$ baskets of food

or *six hundred and twenty-four* items in all! (see Fig. 1.34)

There is a simple reason for this: they had thought of associating easily manipulated material objects with the parts of the body involved in their counting operations. It is true that they counted out the necklaces, pelts and food-baskets by their traditional body-counting method, but the determining element in calculating the ransom to be paid (the number of men lost in the battle) was "numerated" with the help of pebbles and a bundle of sticks.

Let us now imagine that the villagers are working out how to fix the date of an important forthcoming religious festival. The shaman who that morning proclaimed the arrival of the new moon also announced in the following way, accompanying his words with quite precise gestures of his hands, that the festival will fall on the *thirteenth day of the eighth moon thereafter*: "Many suns and many moons will rise and fall before the festival. The moon that has just risen must first wax and then wane completely. Then it must wax as many times again as there are from the little finger on my right hand to the elbow on the same side. Then the sun will rise and set as many times as there are from the little finger on my right hand to my mouth. That is when the sun will next rise on the day of our Great Festival."

This community obviously has a good grasp of the lunar cycle, which is only to be expected, since, after the rising and the setting of the sun, the moon's phases constitute the most obvious regular phenomenon in the natural environment. As in all *empirical calendars*, this one is based on the observation of the first quarter after the end of each cycle. With the help of model collections inherited from forebears, many generations of whom must have contributed to their slow development, the community can in fact "mark time" and compute the date thus expressed without error, as we shall see.

On hearing the shaman's pronouncement, the chief of the tribe paints a number of marks on his own body with some fairly durable kind of colouring material, and these marks enable him to record and to recognise the festival date unambiguously. He first records the series of reappearances that the moon must make from then until the festival by painting *small circles* on his right-hand little finger, ring finger, middle finger, index finger, thumb, wrist, and elbow. Then he records the number of days that must pass from the appearance of the last moon by painting a *thin line*, first of all on each finger of his right hand, then on his right wrist, elbow, shoulder, ear, and eye, then on his nose and mouth. To conclude, he puts a *thick line* over his left eye, thereby symbolising the dawn of the great day itself.

The following day at sunset, a member of the tribe chosen by the chief to "count the moons" takes one of the ready-made ivory tally sticks with thirty incised notches, the sort used whenever it is necessary to reckon the days of a given moon in their order of succession (see Fig. 1.35). He ties a piece of string around the first notch. The next evening, he ties a piece of string around the second notch, and so on every evening until the end of the moon. When he reaches the penultimate notch, he looks carefully at the night sky, in the region where the sun has just set, for he knows that the new moon is soon due to appear.

On that day, however, the first quarter of the new moon is not visible in the sky. So he looks again the next evening when he has tied the string around the last notch on the first tally stick; and though the sky is not clear enough to let him see the new moon, he decides nonetheless that a new month has begun. That is when he paints a little circle on his right little finger, indicating that one lunar cycle has passed.

At dusk the following day, our "moon-counter" takes another similar tally stick and ties a string around the first notch. The day after, he or she proceeds likewise with the second notch, and so on to the end of the second month. But at that month's end the tally man knows he will not need to scan the heavens to check on the rising of the new moon. For in this tribe, the knowledge that moon cycles end alternately on the penultimate and last notches of the tally sticks has been handed down for generations. And this knowledge is only very slightly inaccurate, since the average length of a lunar cycle is 29 days and 12 hours.

FIG. 1.35.

The moon-counter proceeds in this manner through alternating months of 29 and 30 days until the arrival of the last moon, when he paints a little circle on his right elbow. There are now as many circles on the counter's body as on the chief's: the counter's task is over: the "moon tally" has been reached.

The chief now takes over as the "day-counter", but for this task tally sticks are not used, as the body-counting points suffice. The community will celebrate its festival when the chief has crossed out all the *thin lines* from his little finger to his mouth and also the *thick line* over his left eye, that is to say on the thirteenth day of the eighth moon (Fig. 1.36)

This reconstitution of a non-numerate counting system conforms to many of the details observed in Australian aboriginal groups, who are able to reach relatively high numbers through the (unvocalised) numeration of parts of the body when the body-points have a fixed conventional order and are associated with manipulable model collections – knotted string, bundles of sticks, pebbles, notched bones, and so on.

Valuable evidence of this kind of system was reported by Brooke, observing the Dayaks of South Borneo. A messenger had the task of informing a number of defeated rebel villages of the sum of reparations they had to pay to the Dayaks.

> The messenger came along with some dried leaves, which he broke into pieces. Brooke exchanged them for pieces of paper, which were more convenient. The messenger laid the pieces on a table and used his fingers at the same time to count them, up to ten; then he put his foot on the table, and counted them out as he counted out the pieces of paper, each of which corresponded to a village, with the name of its chief, the number of warriors and the sum of the reparation. When he had used up all his toes, he came back to his hands. At the end of the list, there were forty-five pieces of paper laid out on the table.* Then

* Each finger is associated with one piece of paper and one village, in this particular system, and each toe with the set of ten fingers.

he asked me to repeat the message, which I did, whilst he ran through the pieces of paper, his fingers and his toes, as before.

"So there are our letters," he said. "You white folk don't read the way we do."

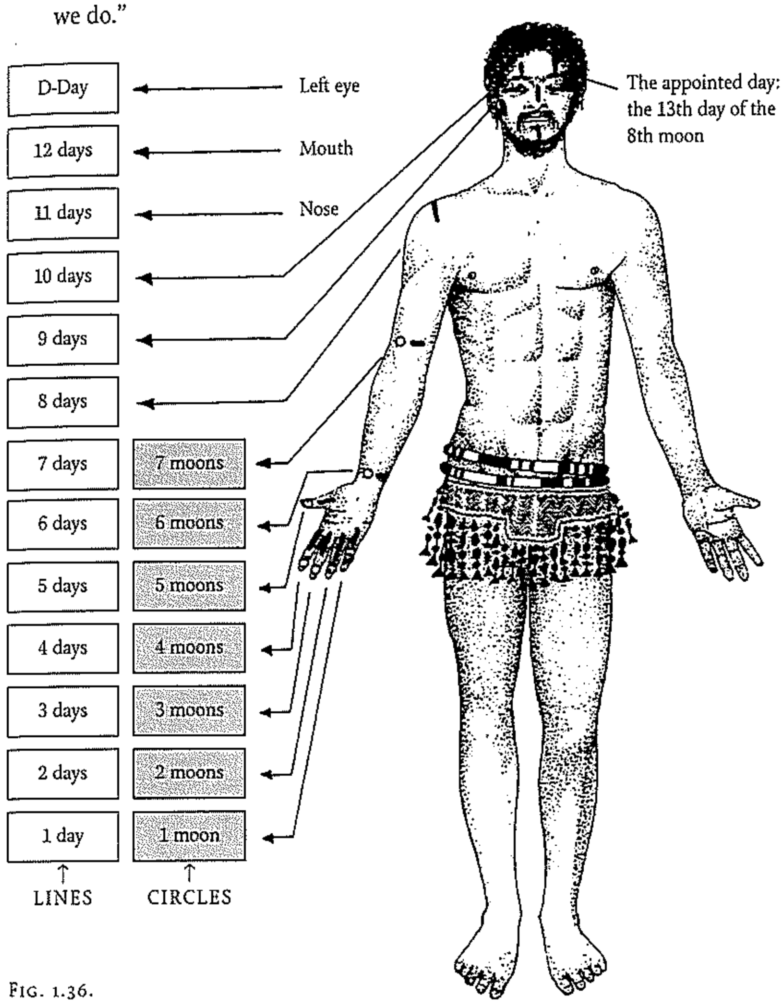


FIG. 1.36.

Later that evening he repeated the whole set correctly, and as he put his finger on each piece of paper in order, he said:

"So, if I remember it tomorrow morning, all will be well; leave the papers on the table."

Then he shuffled them together and made them into a heap. As soon as we got up the next morning, we sat at the table, and he re-sorted the pieces of paper into the order they were in the previous day, and repeated all the details of the message with complete accuracy. For almost a month, as he went from village to village, deep in the interior, he never forgot the different sums demanded. [Adapted from Brooke, *Ten Years in Sarawak*]

All this leads us to hypothesise the following evolution of counting systems:

First stage

Only the lowest numbers are within human grasp. Numerical ability remains restricted to what can be evaluated in a single glance. "Number" is indissociable from the concrete reality of the objects evaluated.* In order to cope with quantities above four, a number of concrete procedures are developed. These include finger-counting and other body-counting systems, all based on one-for-one correspondence, and leading to the development of simple, widely-available ready-made mappings. What is articulated (lexicalised) in the language are these ready-made mappings, accompanied by the appropriate gestures.

Second stage

By force of repetition and habit, the list of the names of the body-parts in their numerative order imperceptibly acquire abstract connotations, especially the first five. They slowly lose their power to suggest the actual parts of the body, becoming progressively more attached to the corresponding number, and may now be applied to any set of objects. (L. Lévy-Bruhl)

Third stage

A fundamental tool emerges: numerical nomenclature, or the names of the numbers.

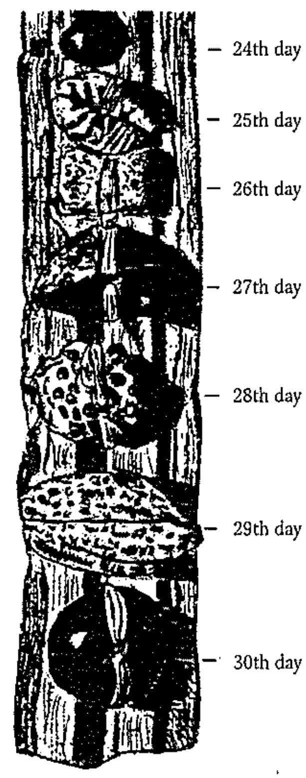


FIG. 1.37. *Detail from a "material model" of a lunar calendar formerly in use amongst tribal populations in former Dahomey (West Africa). It consists of a strip of cloth onto which thirty objects (seeds, kernels, shells, hard fruit, stones, etc.) have been sewn, each standing for one of the days of the month. (The fragment above represents the last seven days). From the Musée de l'Homme, Paris.*

* Thus as L. Lévy-Bruhl reports, Fijians and Solomon islanders have collective nouns for tens of arbitrarily selected items that express neither the number itself nor the objects collected into the set. In Fijian, *bola* means "a hundred dugouts", *koro* "a hundred coconuts", *salavo* "a thousand coconuts". Natives of Mota say *aka peperua* ("butterfly two dugout") for "a pair of dugouts" because of the appearance of the sails. See also Codrington, E. Stephen and L. L. Conant.

COUNTING: A HUMAN FACULTY

The human mind, evidently, can only grasp integers as abstractions if it has fully available to it the notion of distinct units as well as the ability to "synthesise" them. This intellectual faculty (which presupposes above all a complete mastery of the ability to analyse, to compare and to abstract from individual differences) rests on an idea which, alongside mapping and classification, constitutes the starting point of all scientific advance. This creation of the human mind is called "hierarchy relation" or "order relation": it is the principle by which things are ordered according to their "degree of generality", from *individual*, to *kind*, to *type*, to *species*, and so on.

Decisive progress towards the art of abstract calculation that we now use could only be made once it was clearly understood that the integers could be classified into a *hierarchised system of numerical units* whose terms were related as kinds within types, types within species, and so on. Such an organisation of numerical concepts in an invariable sequence is related to the generic principle of "recurrence" to which Aristotle referred (*Metaphysics* 1057, a) when he said that an integer was a "multiplicity measurable by the one". The idea is really that integers are "collections" of abstract units obtained successively by the adjunction of further units.

Any element in the regular sequence of the integers (other than 1) is obtained by adding 1 to the integer immediately preceding in the "natural" sequence that is so constituted (see Fig. 1.38). As the German philosopher Schopenhauer put it, any natural integer presupposes its preceding numbers as the cause of its existence: for our minds cannot conceive of a number as an abstraction unless it subsumes all preceding numbers in the sequence. This is what we called the ability to "synthesise" distinct units. Without that ability, number-concepts remain very cloudy notions indeed.

But once they have been put into a natural sequence, the set of integers permits another faculty to come into play: numeration. To numerate the items in a group is to assign to each a symbol (that is to say, a word, a gesture or a graphic mark) corresponding to a number taken from the natural sequence of integers, beginning with 1 and proceeding in order until the exhaustion of that set (Fig. 1.40). The symbol or name given to each of the elements within the set is the name of its order number within the collection of things, which becomes thereby a sequence or procession of things. The order number of the last element within the ordered group is precisely equivalent to the number of elements in the set. Obviously the number obtained is entirely independent of the order in which the elements are numerated – whichever of the elements you begin with, you always end up with the same total.

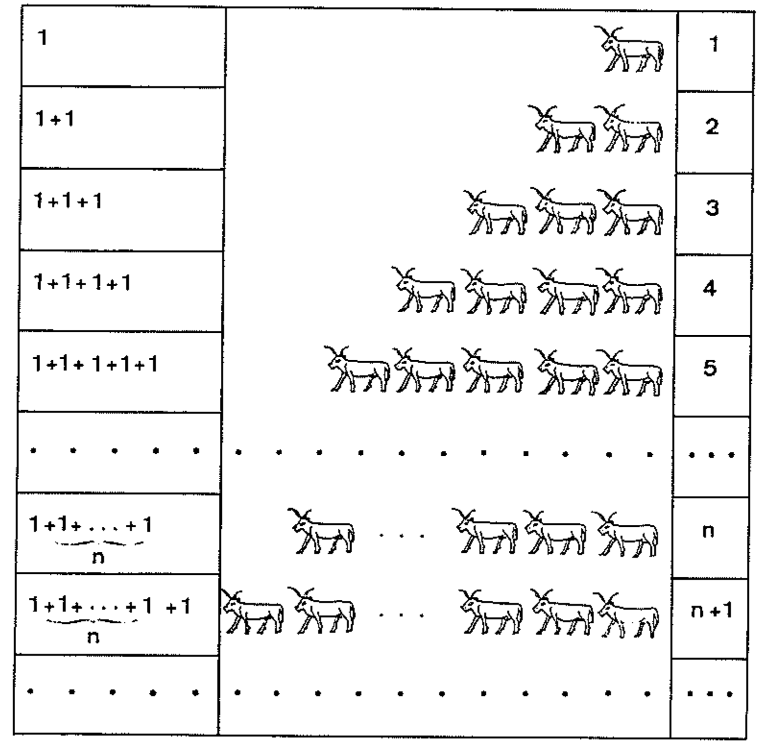


FIG. 1.38. *The generation of integers by the so-called procedure of recurrence*

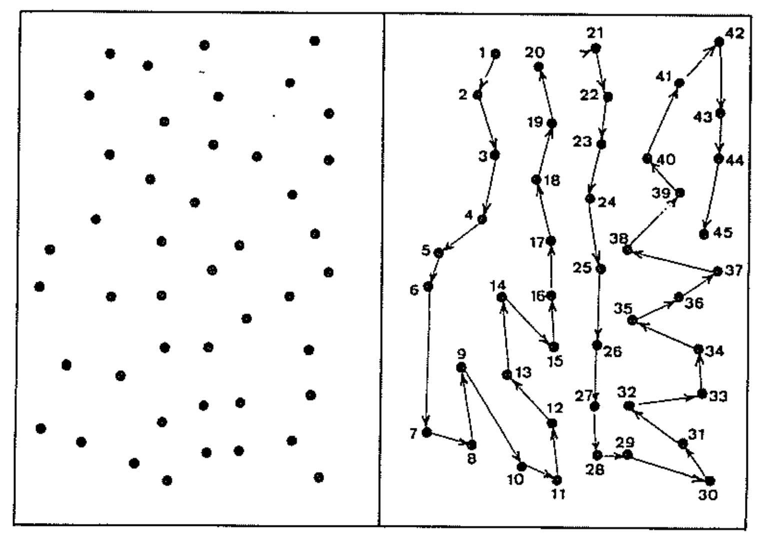


FIG. 1.39. *Numeration of a "cloud" of dots*

For example, let us take a box containing "several" billiard balls. We take out one at random and give it the "number" 1 (for it is the first one to come out of the box). We take another, again completely at random, and give it the "number" 2. We continue in this manner until there are no billiard balls left in the box. When we take out the last of the balls, we give it a specific number from the natural sequence of the integers. If its number is 20, we say that there are "twenty" balls in the box. Numeration has allowed us to transform a vague notion (that there are "several" billiard balls) into exact knowledge.

In like manner, let us consider a set of "scattered" points, in other words dots in a "disordered set" (Fig. 1.39). To find out how many dots there are,

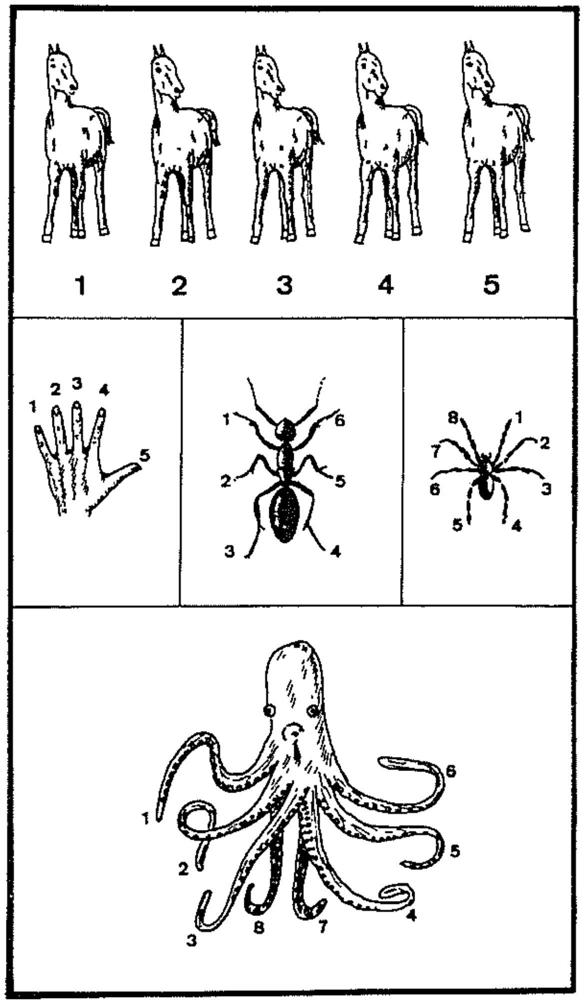


FIG. 1.40. *Numeration allowing us to advance from concrete plurality to abstract number*

all we have to do is to connect them by a "zigzag" line passing through each dot once and no dot twice. The points then constitute what is commonly called a chain. We then give each point in the chain an order-number, starting from one of the ends of the chain we have just made. The last number, given therefore to the last point in the chain, provides us with the total number of dots in the set.

So with the notions of succession and numeration we can advance from the muddled, vague and heterogeneous apperception of concrete plurality to the abstract and homogenous idea of "absolute quantity".

So the human mind can only "count" the elements in a set if it is in possession of all three of the following abilities:

1. the ability to assign a "rank-order" to each element in a procession;
2. the ability to insert into each unit of the procession the memory of all those that have gone past before;
3. the ability to convert a sequence into a "stationary" vision.

The concept of number, which at first sight seemed quite elementary, thus turns out to be much more complicated than that. To underline this point I should like to repeat one of P. Bourdin's anecdotes, as quoted in R. Balmès (1965):

> I once knew someone who heard the bells ring four as he was trying to go to sleep and who counted them out in his head, one, one, one, one. Struck by the absurdity of counting in this way, he sat up and shouted: "The clock has gone mad, it's struck one o'clock four times over!"

THE TWO SIDES OF THE INTEGERS

The concept of number has two complementary aspects: cardinal numbering, which relies only on the principle of mapping, and ordinal numeration, which requires both the technique of pairing and the idea of succession.

Here is a simple way of grasping the difference. January has 31 days. The number 31 represents the total number of days in the month, and is thus in this expression a cardinal number. However, in expressions such as "31 January 1996", the number 31 is not being used in its cardinal aspect (despite the terminology of grammar books) because here it means something like "the thirty-first day" of the month of January, specifying not a total, but a rank-order of a specific (in this case, the last) element in a set containing 31 elements. It is therefore unambiguously an ordinal number.

> We have learned to pass with such facility from cardinal to ordinal number that the two aspects appear to us as one. To determine the plurality of a collection, i.e. its cardinal number, we do not bother any more to find a model collection with which we can match it – we *count*

it. And to the fact that we have learned to identify the two aspects of number is due our progress in mathematics. For whereas in practice we are really interested in the cardinal number, this latter is incapable of creating an arithmetic. The operations of arithmetic are based on the tacit assumption that *we can always pass from any number to its successor*, and this is the essence of the ordinal concept.

And so matching by itself is incapable of creating an art of reckoning. Without our ability to arrange things in ordered succession little progress could have been made. Correspondence and succession, the two principles which permeate all mathematics – nay, all realms of exact thought – are woven into the very fabric of our number-system. [T. Dantzig (1930)]

TEN FINGERS TO COUNT BY

Humankind slowly acquired all the necessary intellectual equipment thanks to the ten fingers on its hands. It is surely no coincidence if children still learn to count with their fingers – and adults too often have recourse to them to clarify their meaning.

Traces of the anthropomorphic origin of counting systems can be found in many languages. In the Ali language (Central Africa), for example, "five" and "ten" are respectively *moro* and *mbouna*: *moro* is actually the word for "hand" and *mbouna* is a contraction of *moro* ("five") and *bouna*, meaning "two" (thus "ten" = "two hands").

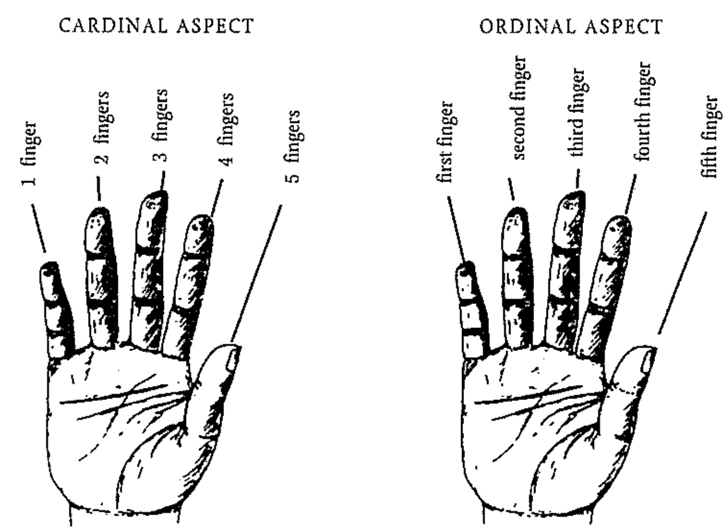


FIG. 1.41.

It is therefore very probable that the Indo-European, Semitic and Mongolian words for the first ten numbers derive from expressions related to finger-counting. But this is an unverifiable hypothesis, since the original meanings of the names of the numbers have been lost.

In any case, the human hand is an extremely serviceable tool and constitutes a kind of "natural instrument" well suited for acquiring the first ten numbers and for elementary arithmetic.

Because there are ten fingers and because each can be moved independently of the others, the hand provides the simplest "model collection" that people have always had – so to speak – to hand.

The asymmetric disposition of the fingers puts the hand in harmony with the normal limitation of the human ability to recognise number visually (a limit set at four). As the thumb is set at some distance from the index finger it is easy to treat it as being "in opposition" to the elementary set of four, and makes the first five numbers an entirely natural sequence. Five thus imposes itself as a basic unit of counting, alongside the other natural grouping, ten. And because each of the fingers is actually different from the others, the human hand can be seen as a true succession of abstract units, obtained by the progressive adjunction of one to the preceding units.

In brief, one can say that the hand makes the two complementary aspects of integers entirely intuitive. It serves as an instrument permitting natural movement between cardinal and ordinal numbering. If you need to show that a set contains three, four, seven or ten elements, you raise or bend *simultaneously* three, four, seven or ten fingers, using your hand as cardinal mapping. If you want to count out the same things, then you bend or raise three, four, seven or ten fingers *in succession*, using the hand as an ordinal counting tool (Fig. 1.41).

The human hand can thus be seen as the simplest and most natural counting machine. And that is why it has played such a significant role in the evolution of our numbering system.