"This is beyond doubt the most interesting book on the evolution of mathematics which has ever fallen into my hands."
—ALBERT EINSTEIN

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the language of science

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edited by joseph mazur | foreword by barry mazur

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The MASTERPIECE SCIENCE *Edition*

A PLUME BOOK

CHAPTER 4

The Last Number

"But what has been said once, can always be repeated."

—Zeno of Elea, as quoted by Simplicius

What is there in mathematics that makes it the acknowledged model of the sciences called exact, and the ideal of the newer sciences which have not yet achieved this distinction? It is, indeed, the avowed ambition of the younger investigators at least, in such fields as biology or the social sciences, to develop standards and methods which will permit these to join the ever-growing ranks of sciences which have already accepted the domination of mathematics.

Mathematics is not only the model along the lines of which the exact sciences are striving to design their structure; mathematics is the cement which holds this structure together. A problem, in fact, is not considered solved until the studied phenomenon has been formulated as a mathematical law. Why is it believed that only mathematical processes can lend to observation, experiment, and speculation that precision, that conciseness, that solid certainty which the exact sciences demand?

When we analyze these mathematical processes we find that they rest on the two concepts: Number and Function; that Function itself can in the ultimate be reduced to Number; that the general concept of Number rests in turn on the properties we ascribe to the natural sequence: one, two, three ....

It is then in the properties of the whole numbers that we may hope to find the clue to this implicit faith in the infallibility of mathematical reasoning!

The first practical application of these properties take the form of the elementary operations of arithmetic; *addition, subtraction, multiplication,* and *division* of whole numbers. We learn these operations very early in life and it is not surprising that most of us have completely forgotten the circumstances under which we acquired them. Let us refresh our memory.

We began by memorizing the table 1 + 1 = 2, 1 + 2 = 3, ... We were drilled and drilled until we were able to add up without hesitancy any two numbers up to ten. In the course of this first phase of our instruction, we were taught to observe that 5 + 3 = 3 + 5 and that this was not an accident, but a general rule. Later we learned to express this property of addition in words: the *sum does not depend on the order of its terms.* The mathematician says no more when he states: *addition is a commutative operation,* and writes in symbols:

$$a + b = b + a.$$

We were next shown that (2 + 3) + 4 = 2 + (3 + 4); by this was meant that whereas (2 + 3) + 4 meant that we add 3 to 2 and 4 to the sum, it was really immaterial in what order we added, for the same result would be obtained if to 2 were added the sum of (3 + 4). The mathematician says no more when he states that addition is an *associative* operation, and writes

$$(a + b) + c = a + (b + c)$$

We never attached much importance to these statements. Yet they are fundamental. On them is based the rule for adding larger numbers. The scheme

$$\begin{array}{r} 25 \\ 34 \\ \underline{56} \\ 115 \end{array}$$

is but a compact paraphrase of:

$$25 + 34 + 56 = (20 + 5) + (30 + 4) + (50 + 6) =$$
$$(20 + 30 + 50) + (5 + 4 + 6) = 100 + 15 = 115$$

in which the commutativity and associativity of addition play a fundamental rôle.

We then proceeded to *multiplication.* Again we memorized a long table until we could tell mechanically the product of any two numbers up to ten. We observed that like addition, *multiplication was both associative and commutative.* Not that we used these words, but we implied as much.

There was yet another property which concerned multiplication and addition jointly. The product 7 × (2 + 3) means that seven is to be multiplied by the sum (2 + 3), that is, by 5; but the same result could be obtained by adding the two partial products (7 × 2) and (7 × 3). The mathematician expresses this in the general statement: multiplication is *distributive* with respect to addition, and writes

$$a(b + c) = ab + ac.$$

It is this distributivity which is at the bottom of the scheme which we use in multiplying numbers greater than ten. Indeed, when we analyze the operation

$$\begin{array}{r} 25 \\ \underline{43} \\ 75 \\ \underline{100} \\ 1075 \end{array}$$

we find it but a compact paraphrase of the involved chain of operations in which this distributive property is freely used.

Thus

$$25 \times 43 = (20 + 5) \times (40 + 3) = [(20 + 5) \times 3] + [(20 + 5) \times 40] = (20 \times 3) + (5 \times 3) + (20 \times 40) + (5 \times 40) = 75 + 1000 = 1075$$

Such are the facts which form the basis of the mathematical education of all thinking men, nay of all people who have had any schooling at all. On these facts is built *arithmetic*, the foundation of mathematics, which in turn supports all science pure and applied which in turn is the fertile source of all technical progress.

Later new facts, new ideas, new concepts were added to our mental equipment, but none of these had to our mind the same security, the same solid foundation, as these properties of whole numbers, which we acquired at the tender age of six. This is expressed in the popular saying: It is as obvious as that two and two make four.

We learned these at an age when we were interested in the "how" of things. By the time we were old enough to ask "why," these rules, through constant use, had become such an intimate part of our mental equipment that they were taken for granted.

The individual is supposed to have retraced in his development the evolution of the species to which he belongs. Some such principle governs the growth of the human intellect as well. In the history of mathematics, the "how" always preceded the "why," the technique of the subject preceded its philosophy.

This is particularly true of arithmetic. The counting technique and the rules of reckoning were established facts at the end of the Renaissance period. But the philosophy of number did not come into its own until the last quarter of the nineteenth century.

As we grow older, we find ample opportunity to apply these rules in our daily tasks, and we grow more and more confident of their generality. The strength of arithmetic lies in its *absolute generality.* Its rules admit of no exceptions: they apply to *all numbers.*

All numbers! Everything hangs on this short but so tremendously important word *all.*

There is no mystery about this word, when it is applied to any *finite* class of things or circumstances. When, for instance, we say "all living men," we attach a very definite meaning to it. We can imagine all mankind arranged in an array of some sort: in this array there will be a *first* man, and there will be a *last* man. To be sure, to prove in all rigor a property true of all living men we should prove it for each individual. While we realize that the actual task would involve insurmountable difficulties, these difficulties, we feel, are of a purely *technical* and not of a *conceptual* character. And this is true of any *finite* collection, i.e., of any collection which has a *last* as well as a *first* member, for *any such collection can be exhausted by counting.*

Can we mean the same thing when we say *all numbers?* Here too, the collection can be conceived as an array, and this array will have a first member, the number *one.* But how about the last?

The answer is ready: *There is no last number!* The process of counting cannot conceivably be terminated. *Every number has a successor.* There is an *infinity* of numbers.

But if there be no last number, what do we mean by all numbers, and particularly, what do we mean by *the property of all numbers?* How can we prove such a property: certainly not by testing every individual case, since we know beforehand that we cannot possibly exhaust all cases.

At the very threshold of mathematics we find this *dilemma of infinity,* like the legendary dragon guarding the entrance to the enchanted garden.

What is the source of this concept of infinity, this faith in the inexhaustibility of the counting process? Is it experience? Certainly not! Experience teaches us the finitude of all things, of all human processes. We know that any attempt on our part to exhaust number by counting would only end in our own exhaustion.

Nor can the existence of the infinite be established mathematically, because infinity, the inexhaustibility of the counting process, is a mathematical assumption, *the basic assumption of arithmetic,* on which all mathematics rests. Is it then a supernatural truth, one of those few gifts which the Creator bestowed upon man when he cast him into the universe, naked and ignorant, but free to shift for himself? Or has the concept of infinity grown upon man, grown out, indeed, out of his futile attempts to reach the last number? Is it but a confession of man's impotence to exhaust the universe by number?

"There is a last number, but it is not in the province of man to reach it, for it belongs to the gods." Such is the keynote of most ancient religions. The stars in the heavens, the grains of sand, the drops of the ocean exemplify this *ultra-ultimate* which is beyond the mind of man to reach. "He counted the stars and named them all," says the psalmist of Jehovah. And Moses in invoking the promise of God to his chosen people says: "He who can count the dust of the earth will also count your seed."

> "There are some, King Gelon, who think that the number of the sands is infinite in multitude; and I mean by sand not only that which exists about Syracuse and the rest of Sicily but also that which is found in every region whether inhabited or uninhabited. Again there are some who, without regarding it as infinite, yet think that no number has been named which is great enough to exceed its multitude. And it is clear that they who hold this view, if they imagined a mass made up of sand in other respects as large as the mass of the earth, including in it all the seas and the hollows of the earth filled up to the height equal to that of the highest mountains, would be many times further still from recognizing that any number could be expressed which exceeded the multitude of the sand so taken. But I will try to show you, by means of geometrical proofs which you will be able to follow, that, of the numbers named by me and given in the work which I sent to Zeuxippus, some exceed not only the number of the mass of sand equal in size to the earth filled up in the way described, but also that of a mass equal in size to the universe."
>
> (Archimedes: The Sand Reckoner)

Now this universe of Archimedes was a sphere limited by the fixed stars. This sphere he estimated to be of a diameter equal to 10,000 earth-diameters. Assuming the number of grains of sand which would fill a poppy seed as 10,000, and the diameter of the earth not greater than 10,000 miles (300,000 stadia), he found for the grains of sand that would fill the universe a fabulous number which in our numeration would be expressed in 52 digits. To express this number Archimedes invented a new unit, *the octade,* which corresponded to our 100,000.

The history of the attempts to square the circle will furnish another example. The problem in its original form was to construct by ruler and compass a square of an area equal to that of a given circle. Now, it is possible to construct a square equivalent to an inscribed regular polygon of say 8 sides. On the other hand it is observed that if we increase the number of sides to 16, 32, 64, etc. we shall approximate the area of the circle more and more closely. Now there is no doubt that some of the Greek geometers regarded this doubling process not as an approximation but as a means of attaining the circle, i.e., they thought if they could continue this process long enough they would eventually reach the ultimate polygon which would coincide with the circle at all points.

It is a plausible hypothesis that the early conception of infinity was not the uncountable, but the yet-uncounted. The last number meant *patience* and *perseverance*, and man seemed to be lacking in these qualities. It was of the same order of things as reaching heaven in the story of the Tower of Babel. The last number, like the heavens, belonged to God. In His jealous wrath He would confound the tongues of the ambitious builders.

This confusion of tongues persists to this day. Around infinity have grown up all the paradoxes of mathematics: from the arguments of Zeno to the antimonies of Kant and Cantor. This story we shall tell in another chapter. What concerns us here is that these paradoxes were instrumental in creating a more critical attitude towards the foundations of arithmetic. For, since the properties of whole numbers form the basis of mathematics, if these properties can be proved by the rules of *formal* logic, then all of mathematics is a logical discipline. If, however, logic is insufficient to establish these properties, then mathematics is founded on something more than mere logic: its creative power relies on that elusive, intangible thing which is called human *intuition*.

Let there be no misunderstanding! It is not the validity of these properties of number which is at stake; the issue is the validity of the arguments which purport to prove the validity of these properties. The questions that have been at issue ever since the foundations of mathematics were submitted to this searching analysis, the questions which have split the leading mathematical thinkers into two contending camps, *intuitionists* vs. *formalists*, are these: What constitutes a mathematical proof? What is the nature of reasoning generally and mathematical reasoning in particular? What is meant by *mathematical existence?*

Now, the laws of sound reasoning are as old as the hills. They were formulated in a systematic manner by Aristotle, but were known long before him. Why, they are the very skeleton of the human intellect: every intelligent man has occasion to apply these laws in his daily pursuits. He knows, that in order to reason soundly, he must first define his premises without ambiguity, then through a step-by-step application of the canons of logic he will eventually arrive at a conclusion which is the *unique* consequence of the logical process he used in reaching it.

If this conclusion does not tally with the facts as we observe them, then the first step is to find out whether we applied these canons correctly. This is not the place to analyze the validity of these canons. Not that they have been spared the scorching fire of this critical age! Quite the contrary: one of them is, indeed, the center of a controversy which has been raging for a quarter of a century and which shows no sign of abating. However, this is a story by itself and it will be told in its proper place.

If it is found that the canons of logic were applied correctly, then the discrepancy, if there be a discrepancy, may mean that there is something wrong with our premises. There may be an inconsistency lurking somewhere in our assumptions, or one of our premises may contradict another.

Now, to establish a set of assumptions for any particular body of knowledge is not an easy task. It requires not only acute analytical judgment, but great skill as well. For, in addition to this freedom from contradiction, it is desired that each assumption should be independent of all the others, and that the whole system be exhaustive, i.e., completely cover the question under investigation. The branch of mathematics which deals with such problems is called *axiomatics* and has been cultivated by such men as Peano, Russell and Hilbert. In this manner logic, formerly a branch of philosophy, is being gradually absorbed into the body of mathematics.

Returning to our problem, suppose that we have examined our premises and have found them free from contradictions. Then we say that our conclusion is logically flawless. If, however, this conclusion does not agree with the observed facts, we know that the assumptions we have made do not fit the concrete problem to which they were applied. There is nothing wrong with the tailoring of the suit. If it bulges in some spots and cracks in others, it is the fault of the *fitter.*

The process of reasoning just described is called *deductive.* It consists in starting from very general properties, which take the form of *definitions, postulates* or *axioms,* and in deriving from these, by means of the canons of logic, statements concerning things or circumstances which would occur in particular instances.

The process of deduction is characteristic of mathematical reasoning. It has found a nearly complete realization in geometry, and for this reason the logical structure of geometry has been the model for all exact sciences.

Quite different in its nature is the other method used in scientific investigation: *induction.* It is generally described as

proceeding from the particular to the general. It is the result of observation and experience. To discover a property of a certain class of objects we repeat the observation or tests as many times as feasible, and under circumstances as nearly similar as possible. Then it may happen that a certain definite tendency will manifest itself throughout our observation or experimentation. This tendency is then accepted as the property of the class. For example, if we subject a sufficiently large number of samples of lead to the action of heat, and we find that in every case melting began when the thermometer reached 328°, we conclude that the point of fusion of lead is 328°. Back of this is the conviction that no matter how many more samples we might test, the circumstances not having changed, the results would also be the same.

This process of induction, which is basic in all experimental sciences, is *for ever banned* from rigorous mathematics. Not only would such a proof of a mathematical proposition be considered ridiculous, but even as a verification of an established truth it would be inacceptable. *For, in order to prove a mathematical proposition, the evidence of any number of cases would be insufficient, whereas to disprove a statement one example will suffice.* A mathematical proposition is true, if it leads to no logical contradiction, false otherwise. *The method of deduction is based on the principle of contradiction and on nothing else.*

Induction is barred from mathematics and for a good reason. Consider the quadratic expression $(n^2 - n + 41)$ which I mentioned in the preceding chapter. We set in this expression $n = 1, 2, 3$ ......up to $n = 40$: in each of these cases we get a prime number as the result. Shall we conclude that this expression represents prime numbers for all values of n? Even the least mathematically trained reader will recognize the fallacy of such a conclusion: yet many a physical law has been held valid on less evidence.

Mathematics is a deductive science, arithmetic is a branch of mathematics. Induction is inadmissible. The propositions of arithmetic, the associative, commutative and distributive properties of the operations, for instance, which play such a fundamental rôle even in the most simple calculations, must be demonstrated by deductive methods. What is the principle involved?

Well, this principle has been variously called *mathematical induction,* and *complete induction,* and that of *reasoning by recurrence.* The latter is the only acceptable name, the others being misnomers. The term induction conveys an entirely erroneous idea of the method, for it does not imply systematic trials.

To give an illustration from a familiar field, let us imagine a line of soldiers. Each one is instructed to convey any information that he may have obtained to his neighbor on the right. The commanding officer who has just entered the field wants to ascertain whether *all* the soldiers know of a certain event that has happened. Must he inquire of every soldier? Not if he is sure that whatever any soldier may know his neighbor to the right is also bound to know, for then if he has ascertained that the *first* soldier to the left knows of the event he can conclude that *all* the soldiers know of it.

The argument used here is an example of reasoning by recurrence. It involves two stages. It is first shown that the proposition we wish to demonstrate is of the type which Bertrand Russell calls *hereditary:* i.e., if the proposition were true for any member of a sequence, its truth for the *successor* of the member would follow as a logical necessity. In the second place, it is shown that the proposition is true for the first term of the

sequence. This latter is the so-called *induction* step. Now in view of its hereditary nature, the proposition, being true of the first term, must be true of the second, and being true of the second it must be true of the third, etc., etc. We continue in this way till we have exhausted the whole sequence, i.e., reached its *last* member.

Both steps in the proof, the induction and the hereditary feature, are necessary; neither is sufficient alone. The history of the two theorems of Fermat may serve as illustration. The first theorem concerns the statement that $2^{2^n} + 1$ is a prime for all values of n. Fermat showed by actual trial that such is the case for $n = 0, 1, 2, 3$ or 4. But he could not prove the hereditary property; and as a matter of fact, we saw that Euler disproved the proposition by showing that it fails for $n = 5$. The second theorem alleges that the equation $x^n + y^n = z^n$ cannot be solved in integers when n is greater than 2. Here the induction step would consist in showing that the proposition holds for n = 3, i.e., that the equation $x^3 + y^3 = z^3$ cannot be solved in whole numbers. It is possible that Fermat had a proof of this, and if so, here would be one interpretation of the famous marginal note. At any rate, this first step, we saw, was achieved by Euler. It remains to show that the property is hereditary, i.e., assuming it true for some value of n, say p, it should follow as a logical necessity that the equation $x^{p+1} + y^{p+1} = z^{p+1}$ cannot be solved in integers.

It is significant that we owe the first explicit formulation of the *principle of recurrence* to the genius of Blaise Pascal, a contemporary and friend of Fermat. Pascal stated the principle in a tract called *The Arithmetic Triangle* which appeared in 1654. Yet it was later discovered that the gist of this tract was contained in the correspondence between Pascal and Fermat regarding a problem in gambling, the same correspondence which is now regarded as the nucleus from which developed the theory of probabilities.

It surely is a fitting subject for mystic contemplation, that the principle of reasoning by recurrence, which is so basic in pure mathematics, and the theory of probabilities, which is the basis of all inductive sciences, were both conceived while devising a scheme for the division of the stakes in an unfinished match of two gamblers.

How the principle of mathematical induction applies to Arithmetic can be best illustrated in the proof that addition of whole numbers is an *associative* operation. In symbols this means:

(1) $$a + (b + c) = (a + b) + c$$

Let us analyze the operation $a + b$: it means that to the number a was added 1, to the result was added 1 again, and that this process was performed b times. Similarly $a + (b + 1)$ means $b + 1$ successive additions of 1 to a. It follows therefore that:

(2) $$a + (b + 1) = (a + b) + 1$$

and this is proposition (1) for the case when $c = 1$. What we have done, so far, constitutes the *induction* step of our proof.

Now for the hereditary feature. Let us assume that the proposition is true for some value of c, say n, i.e.

(3) $$a + (b + n) = (a + b) + n$$

Adding 1 to both sides:

(4) $$[a + (b + n)] + 1 = [(a + b) + n] + 1$$

which because of (2) can be written as

(5) $$(a + b) + (n + 1) = a + [(b + n) + 1]$$

And for the same reason is equivalent to

(6) $$(a + b) + (n + 1) = a + [b + (n + 1)]$$

but this is proposition (1) for the case $c = n + 1$.

Thus the fact that the proposition is true for some number n carries with it *as a logical necessity* that it must be true for the successor of that number, $n + 1$. Being true for 1, it is therefore true for 2; being true for 2, it is true for 3; and so on *indefinitely.*

The principle of mathematical induction in the more general form in which it is here applied can be formulated as follows: Knowing that a proposition involving a sequence is true for the first number of the sequence, and that the assumption of its truth for some particular member of the sequence involves as a logical consequence the truth of the proposition for the successor of the number, we conclude that it is true for all the numbers of the sequence. The difference between the *restricted* principle as it was used in the case of the soldiers, and the *general* principle as it is used in arithmetic, is merely in the interpretation of the word *all.*

Let me repeat: it is not by means of the restricted, but of the general principle of mathematical induction that the validity of the operations of arithmetic which we took on faith when we were first initiated into the mysteries of number has been established.

The excerpts in the following section are taken from an article by Henri Poincaré entitled *The Nature of Mathematical Reasoning.* This epoch-making essay appeared in 1894 as the first of a series of investigations into the foundations of the exact sciences. It was a signal for a throng of other mathematicians to inaugurate a movement for the revision of the classical concepts, a movement which culminated in the nearly complete absorption of logic into the body of mathematics.

The great authority of Poincaré, the beauty of his style, and the daring iconoclasm of his ideas carried his work far beyond the limited public of mathematicians. Some of his biographers estimated that his writing reached half a million people, an audience which no mathematician before him had ever commanded.

Himself a creator in practically every branch of mathematics, physics, and celestial mechanics, he was endowed with a tremendous power of introspection which enabled him to analyze the sources of his own achievements. His penetrating mind was particularly interested in the most elementary concepts, concepts which the thick crust of human habit has made almost impenetrable: to these concepts belong *number*, *space*, and *time*.

"The very possibility of a science of mathematics seems an insoluble contradiction. If this science is deductive only in appearance, whence does it derive that perfect rigor which no one dares to doubt? If, on the contrary, all the propositions it enunciates can be deduced one from the other by the rules of formal logic, why is not mathematics reduced to an immense tautology? The syllogism can teach us nothing that is essentially new, and, if everything is to spring from the principle of identity, everything should be capable of being reduced to it. Shall we then admit that the theorems which fill so many volumes are nothing but devious ways of saying that A is A?

"We can, no doubt, fall back on the axioms, which are the source of all these reasonings. If we decide that these cannot be reduced to the principle of contradiction, if still less we see in them experimental facts, ... we have yet the resource of regarding them as *a priori* judgments. This will not solve the difficulty but only christen it ....

"The rule of reasoning by recurrence is not reducible to the principle of contradiction. ... Nor can this rule come to us from experience. Experience could teach us that the rule is true for the first ten or hundred numbers; it cannot attain the indefinite series of numbers, but only a portion of this series, more or less long, but always limited.

"Now, if it were only a question of a portion, the principle of contradiction would suffice; it would always allow of our developing as many syllogisms as we wished. It is only when it is a question of including an infinity of them in a single formula, it is only before the infinite, that this principle of logic fails, and here is where experience too becomes powerless ....

"Why then does this judgment force itself upon us with such an irresistible force? It is because it is only the affirmation of the power of the mind which knows itself capable of conceiving the indefinite repetition of the same act when this act is possible at all ....

"There is, we must admit, a striking analogy between this and the usual procedure of induction. But there is an essential difference. Induction, as applied in the physical sciences, is always uncertain, because it rests on the belief in a general order in the universe, an order outside of us. On the contrary, mathematical induction, i.e., demonstration by recurrence, imposes itself as a necessity, because it is only a property of the mind itself ....

"We can ascend only by mathematical induction, which alone can teach us something new. Without the aid of this induction, different from physical induction but just as fertile, deduction would be powerless to create a science.

"Observe, finally, that this induction is possible only if the same operation can be repeated indefinitely. That is why the theory of chess can never become a science: the different moves of the game do not resemble one another."

The last word should go to the master and so I should have liked to conclude this chapter. But history is no respecter of persons: the ideas of Poincaré raised a controversy which rages to this day. And so I must add a word of my own, not in the hope of contributing something to the issues which have been so exhaustively treated by the eminent men on both sides of the question, but in order that the true issue may be brought out in relief.

Reasoning by recurrence, whenever it is applied to finite sequences of numbers, is logically unassailable. In this *restricted* sense, the principle asserts that, if a proposition is of the hereditary type, then it is true or false of any term in the sequence if it is true or false of the first term in the sequence.

This *restricted principle* will suffice to create a finite, bounded arithmetic. For instance, we could terminate the natural sequence at the physiological or psychological limits of the counting process, say 1,000,000. In such an arithmetic addition and multiplication, when possible, would be *associative* and *commutative*; but the operations would not always be possible. Such expressions as (500,000 + 500,001) or (1000 × 1001) would be meaningless, and it is obvious that the number of meaningless cases would far exceed those which have a meaning. This restriction on integers would cause a corresponding restriction on fractions; no decimal fraction could have more than 6 places, and the conversion of such a fraction as 1/3 into a decimal fraction would have no meaning. Indefinite divisibility would have no more meaning than indefinite growth, and we would reach the indivisible by dividing any object into a million equal parts.

A similar situation would arise in geometry if instead of conceiving the plane as indefinitely extending in all direction we should limit ourselves to a *bounded region* of the plane, say a circle. In such a bounded geometry the intersection of two lines would be a matter of probability; two lines taken at random would not determine an angle; and three lines taken at random would not determine a triangle.

Yet, not only would such a bounded arithmetic and such a bounded geometry be logically impregnable, but strange though

it may seem at first, they would be closer to the reality of our senses than are the unbounded varieties which are the heritage of the human race.

The restricted principle of mathematical induction involves a finite chain of syllogisms, each consistent in itself: for this reason the principle is a consequence of classical logic.

But the method used in the demonstrations of arithmetic, the *general* principle of complete induction, goes far beyond the confines imposed by the restricted principle. It is not content to say that a proposition true for the number 1 is true for all numbers, provided that if true for any number it is true for the successor of this number. *It tacitly asserts that any number has a successor.*

This assertion is not a logical necessity, for it is not a consequence of the laws of classical logic. This assertion does *not* impose itself as the only one conceivable, for its opposite, the postulation of a finite series of numbers, leads to a bounded arithmetic which is just as tenable. This assertion is *not* derived from the immediate experience of our senses, for all our experience proclaims its falsity. And finally this assertion is *not* a consequence of the historical development of the experimental sciences, for all the latest evidence points to a bounded universe, and in the light of the latest discoveries in the structure of the atom, the infinite divisibility of matter must be declared a myth.

And yet the concept of infinity, though not imposed upon us either by logic or by experience, is a *mathematical necessity.* What is, then, behind this power of the mind to conceive the indefinite repetition of an act when this act is once possible? To this question I shall return again and again throughout this study.