"This is beyond doubt the most interesting book on the evolution of mathematics which has ever fallen into my hands."
—ALBERT EINSTEIN

number

the language of science

tobias dantzig

edited by joseph mazur | foreword by barry mazur

TOBIAS DANTZIG

NUMBER

The Language of Science

Edited by
JOSEPH MAZUR

Foreword by
BARRY MAZUR

The MASTERPIECE SCIENCE *Edition*

A PLUME BOOK

NUMBER WORDS OF SOME INDO-EUROPEAN LANGAUGES SHOWING THE EXTRAORDINARY STABILITY OF NUMBER WORDS

	Sanskrit	Ancient Greek	Latin	German	English	French	Russian
1	eka	en	unus	eins	one	un	odyn
2	dva	duo	duo	zwei	two	deux	dva
3	tri	tri	tres	drei	three	trois	tri
4	catur	tetra	quatuor	vier	four	quatre	chetyre
5	panca	pente	quinque	fünf	five	cinq	piat
6	sas	hex	sex	sechs	six	six	shest
7	sapta	hepta	septem	sieben	seven	sept	sem
8	asta	octo	octo	acht	eight	huit	vosem
9	nava	ennea	novem	neun	nine	neuf	deviat
10	daca	deca	decem	zehn	ten	dix	desiat
100	cata	ecaton	centum	hundert	hundred	cent	sto
1000	sehastre	xilia	mille	tausend	thousand	mille	tysiaca

A TYPICAL QUINARY SYSTEM: THE API LANGUAGE OF THE NEW HEBRIDES

	Word	Meaning
1	tai	
2	lua	
3	tolu	
4	vari	
5	luna	hand
6	otai	other one
7	olua	" two
8	otolu	" three
9	ovair	" four
10	lua luna	two hands

A TYPICAL VIGESIMAL SYSTEM: THE MAYA LANGUAGE OF CENTRAL AMERICA

1	hun	1
20	kal	20
20^2	bak	400
20^3	pic	8000
20^4	calab	160,000
20^5	kinchel	3,200,000
20^6	alce	64,000,000

A TYPICAL BINARY SYSTEM: A WESTERN TRIBE OF TORRES STRAITS

1 urapun 2 okosa	3 okosa-urapun 4 okosa-okosa	5 okosa-okosa-urapun 6 okosa-okosa-okosa

CHAPTER 2

The Empty Column

"It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value; a profound and important idea which appears so simple to us now that we ignore its true merit. But its very simplicity and the great ease which it has lent to all computations put our arithmetic in the first rank of useful inventions; and we shall appreciate the grandeur of this achievement the more when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity."

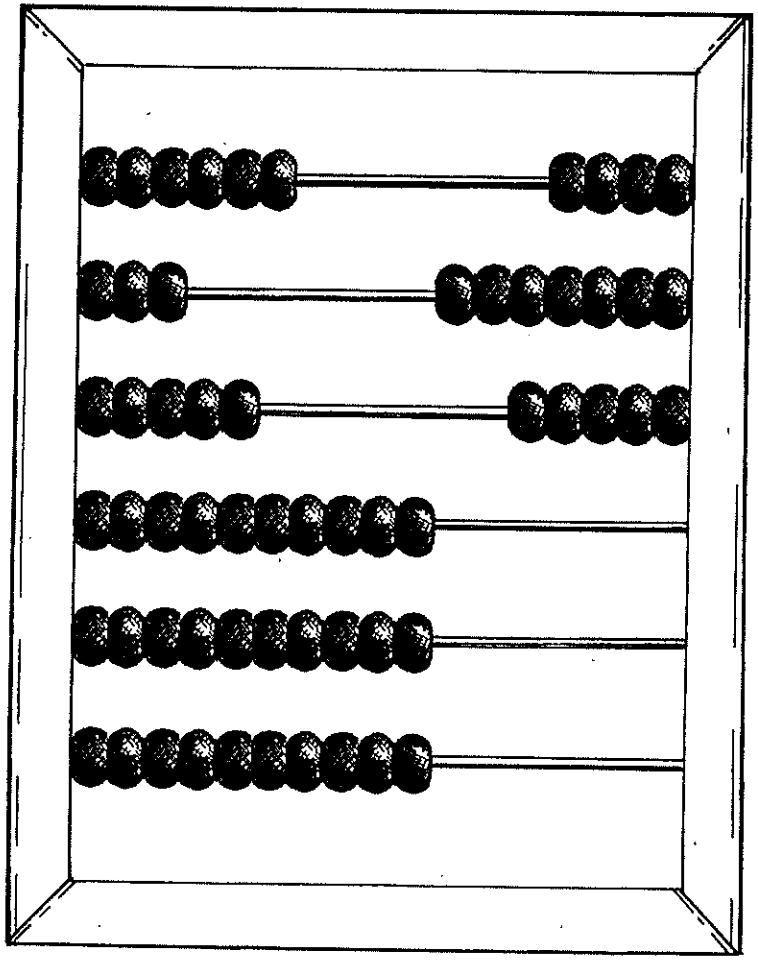
—Laplace

As I am writing these lines there rings in my ears the old refrain:

"Reading, 'Riting, 'Rithmetic,
Taught to the tune of a hickory-stick!"

In this chapter I propose to tell the story of one of three R's, the one, which, though oldest, came hardest to mankind.

It is not a story of brilliant achievement, heroic deeds, or noble sacrifice. It is a story of blind stumbling and chance discovery, of groping in the dark and refusing to admit the light. It is a story replete with obscurantism and prejudice, of sound judgment often eclipsed by loyalty to tradition, and of reason long held subservient to custom. In short, it is a human story.

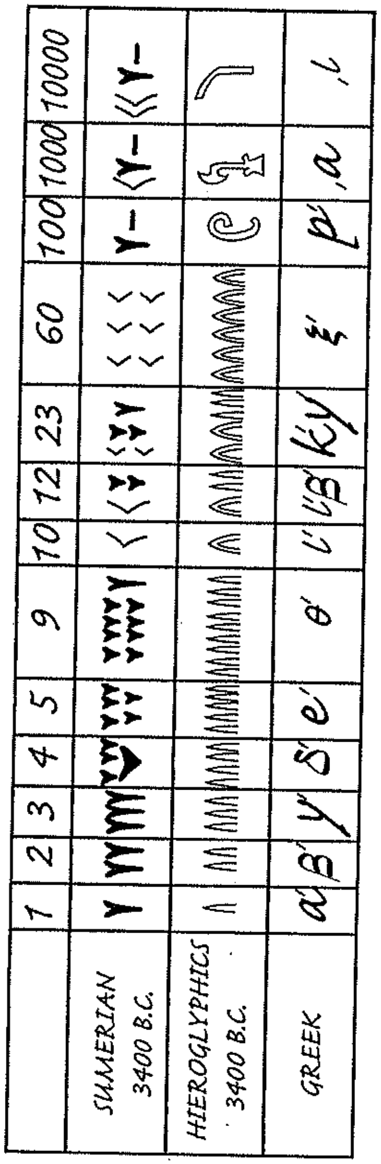


A SCHEMATIC DRAWING OF A COUNTING BOARD

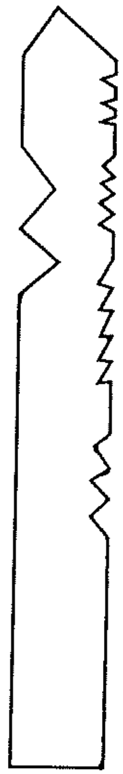
Written numeration is probably as old as private property. There is little doubt that it originated in man's desire to keep a record of his flocks and other goods. Notches on a stick or tree, scratches on stones and rocks, marks in clay—these are the earliest forms of this endeavor to record numbers by written symbols. Archeological researches trace such records to times immemorial, as they are found in the caves of prehistoric man in Europe, Africa and Asia. Numeration is at least as old as written language, and there is evidence that it preceded it. Perhaps, even, the recording of numbers had suggested the recording of sounds.

The oldest records indicating the systematic use of written numerals are those of the ancient Sumerians and Egyptians. They are all traced back to about the same epoch, around 3500 B.C. When we examine them we are struck with the great similarity in the principles used. There is, of course, the possibility that there was communication between these peoples in spite of the distances that separated them. However, it is more likely that they developed their numerations along the lines of least resistance, i.e., that their numerations were but an outgrowth of the natural process of tallying. (See figure page 22.)

Indeed, whether it be the cuneiform numerals of the ancient Babylonians, the hieroglyphics of the Egyptian papyri, or the queer figures of the early Chinese records, we find everywhere a distinctly *cardinal* principle. Each numeral up to nine is merely a collection of strokes. The same principle is used beyond nine, units of a higher class, such as tens, hundreds, etc., being represented by special symbols.



ANCIENT NUMERATIONS



SCHEMATIC DRAWING OF ENGLISH TALLY-STICK

The English tally-stick of obscure but probably very ancient origin, also bears this unquestionably cardinal character. A schematic picture of the tally is shown in the accompanying figure. The small notches each represent a pound sterling, the larger ones 10 pounds, 100 pounds, etc.

It is curious that the English tally persisted for many centuries after the introduction of modern numeration made its use ridiculously obsolete. In fact it was responsible for an important episode in the history of Parliament. Charles Dickens described this episode with inimitable sarcasm in an address on Administrative Reform, which he delivered a few years after the incident occurred.

"Ages ago a savage mode of keeping accounts on notched sticks was introduced into the Court of Exchequer and the accounts were kept much as Robinson Crusoe kept his calendar on the desert island. A multitude of accountants, bookkeepers, and actuaries were born and died .... Still official routine inclined to those notched sticks as if they were pillars of the Constitution, and still the Exchequer accounts continued to be kept on certain splints of elm-wood called *tallies.* In the reign of George III an inquiry was made by some revolutionary spirit whether, pens, ink and paper, slates and pencils being in existence, this obstinate adherence to an obsolete custom ought to be continued, and whether a change ought not be effected. All the red tape in the country grew redder at the bare mention of this bold and original conception, and it took until 1826 to get these sticks abolished. In 1834 it was found that there was a considerable accumulation of them; and the question then arose, what was to be done with such worn-out, worm-eaten, rotten old bits of wood? The sticks were housed in Westminster, and it would naturally occur to any intelligent person that nothing could be easier than to allow them to be carried away for firewood by the miserable people who lived in that neighborhood. However, they never had been useful, and official routine required that they should never be, and so the order went out that they were to be privately and confidentially burned. It came to pass that they were burned in a stove in the House of Lords. The stove, over-gorged with these preposterous sticks, set fire to the panelling; the panelling set fire to the House of Commons; the two houses were reduced to ashes; architects were called in to build others; and we are now in the second million of the cost thereof."

As opposed to this purely cardinal character of the earliest records there is the ordinal numeration, in which the numbers are represented by the letters of an alphabet in their spoken succession.

The earliest evidence of this principle is that of the Phoenician numeration. It probably arose from the urge for compactness brought about by the complexities of a growing commerce. The Phoenician origin of both the Hebrew and the Greek numeration is unquestionable: the Phoenician system was adopted bodily, together with the alphabet, and even the sounds of the letters were retained.

On the other hand, the Roman numeration, which has survived to this day, shows a marked return to the earlier cardinal methods. Yet Greek influence is shown in the literal symbols adopted for certain units, such as X for ten, C for hundred, M for thousand. But the substitution of letters for the more picturesque symbols of the Chaldeans or the Egyptians does not constitute a departure from principle.

The evolution of the numerations of antiquity found its final expression in the ordinal system of the Greeks and the cardinal

system of Rome. Which of the two was superior? The question would have significance if the only object of a numeration were a compact recording of quantity. But this is not the main issue. A far more important question is: how well is the system adapted to arithmetical operations, and what ease does it lend to calculations?

In this respect there is hardly any choice between the two methods: neither was capable of creating an arithmetic which could be used by a man of average intelligence. This is why, from the beginning of history until the advent of our modern *positional* numeration, so little progress was made in the art of reckoning.

Not that there were no attempts to devise rules for operating on these numerals. How difficult these rules were can be gleaned from the great awe in which all reckoning was held in these days. A man skilled in the art was regarded as endowed with almost supernatural powers. This may explain why arithmetic from time immemorial was so assiduously cultivated by the priesthood. We shall have occasion later to dwell at greater length on this relation of early mathematics to religious rites and mysteries. Not only was this true of the ancient Orient, where science was built around religion, but even the enlightened Greeks never completely freed themselves from this mysticism of number and form.

And to a certain extent this awe persists to this day. The average man identifies mathematical ability with quickness in figures. "So you are a mathematician? Why, then you have no trouble with your income-tax return!" What mathematician has not at least once in his career been so addressed? There is, perhaps, unconscious irony in these words, for are not most professional mathematicians spared all trouble incident to excessive income?

There is a story of a German merchant of the fifteenth century, which I have not succeeded in authenticating, but it is so characteristic of the situation then existing that I cannot resist the temptation of telling it. It appears that the merchant had a son whom he desired to give an advanced commercial education. He appealed to a prominent professor of a university for advice as to where he should send his son. The reply was that if the mathematical curriculum of the young man was to be confined to adding and subtracting, he perhaps could obtain the instruction in a German university; but the art of multiplying and dividing, he continued, had been greatly developed in Italy, which in his opinion was the only country where such advanced instruction could be obtained.

As a matter of fact, multiplication and division as practiced in those days had little in common with the modern operations bearing the same names. Multiplication, for instance, was a succession of *duplations*, which was the name given to the doubling of a number. In the same way division was reduced to *mediation*, i.e., "halving" a number. A clearer insight into the status of reckoning in the Middle Ages can be obtained from an example. Using modern notations:

Today	Thirteenth century
46	$46 \times 2 = 92$
13	$46 \times 4 = 92 \times 2 = 184$
138	$46 \times 8 = 184 \times 2 = 368$
46	$368 + 184 + 46 = 598$
598	

We begin to understand why humanity so obstinately clung to such devices as the abacus or even the tally. Computations which a child can now perform required then the services of a specialist, and what is now only a matter of a few minutes meant in the twelfth century days of elaborate work.

The greatly increased facility with which the average man today manipulates number has been often taken as proof of the growth of the human intellect. The truth of the matter is that the difficulties then experienced were inherent in the numeration in use, a numeration not susceptible to simple, clear-cut rules. The discovery of the modern positional numeration did away with these obstacles and made arithmetic accessible even to the dullest mind.

The growing complexities of life, industry and commerce, of landed property and slave-holding, of taxation and military organization—all called for calculations more or less intricate, but beyond the scope of the finger technique. The rigid, unwieldy numeration was incapable of meeting the demand. How did man, in the five thousand years of his civilized existence which preceded modern numeration, counter these difficulties?

The answer is that from the very outset he had to resort to mechanical devices which vary in form with place and age but are all the same in principle. The scheme can be typified by the curious method of counting an army which has been found in Madagascar. The soldiers are made to file through a narrow passage, and one pebble is dropped for each. When 10 pebbles are counted, a pebble is cast into another pile representing tens, and the counting continues. When 10 pebbles are amassed in the second pile, a pebble is cast into a third pile representing hundreds, and so on until all the soldiers have been accounted for.

From this there is but one step to the *counting board* or *abacus* which in one form or another has been found in practically every country where a counting technique exists. The abacus in its general form consists of a flat board divided into a series of parallel columns, each column representing a distinct decimal class, such as units, tens, hundreds, etc. The board is provided with a set of counters which are used to indicate the number of units in each class. For instance, to represent 574 on the abacus, 4 counters are put on the last column, 7 counters on the next to last and 5 on the third to the last column. (See figure, page 20.)

The many counting boards known differ merely in the construction of the columns and in the type of counters used. The Greek and Roman types had loose counters, while the Chinese Suan-Pan of today has perforated balls sliding on slender bamboo rods. The Russian Szczety, like the Chinese variety, consists of a wooden frame on which are mounted a series of wire rods with sliding buttons for counters. Finally, it is more than probable that the ancient Hindu *dust board* was also an abacus in principle, the part of the counters here being played by erasable marks written on sand.

The origin of the word abacus is not certain. Some trace it to the Semitic *abac,* dust; others believe that it came from the Greek *abax,* slab. The instrument was widely used in Greece, and we find references to it in Herodotus and Polybius. The latter, commenting on the court of Philip II of Macedonia in his *Historia* makes this suggestive statement:

> "Like counters on the abacus which at the pleasure of the calculator may at one moment be worth a talent and the next moment a chalcus, so are the courtiers at their King's nod at one moment at the height of prosperity and at the next objects of human pity."

To this day the counting board is in daily use in the rural districts of Russia and throughout China, where it persists in open competition with modern calculating devices. But in Western Europe and America the abacus survived as a mere curiosity which few people have seen except in pictures. Few realize how extensively the abacus was used in their own countries only a few hundred years ago, where after a fashion it managed to meet the difficulties which were beyond the power of a clumsy numeration.

One who reflects upon the history of reckoning up to the invention of the principle of position is struck by the paucity of achievement. This long period of nearly five thousand years saw the fall and rise of many a civilization, each leaving behind it a heritage of literature, art, philosophy and religion. But what was the net achievement in the field of reckoning, the earliest art practiced by man? An inflexible numeration so crude as to make progress well-nigh impossible, and a calculating device so limited in scope that even elementary calculations called for the services of an expert. And what is more, man used these devices for thousands of years without making a single worth-while improvement in the instrument, without contributing a single important idea to the system!

This criticism may sound severe; after all it is not fair to judge the achievements of a remote age by the standards of our own time of accelerated progress and feverish activity. Yet, even when compared with the slow growth of ideas during the Dark Ages, the history of reckoning presents a peculiar picture of desolate stagnation.

When viewed in this light, the achievement of the unknown Hindu who some time in the first centuries of our era discovered the *principle of position* assumes the proportions of a world-event. Not only did this principle constitute a radical departure in method, but we know now that without it no progress in arithmetic was possible. And yet the principle is so simple that today the dullest school boy has no difficulty in grasping it. In a measure, it is suggested by the very structure of our number language. Indeed, it would appear that the first attempt to translate the action of the counting board into the language of numerals ought to have resulted in the discovery of the principle of position.

Particularly puzzling to us is the fact that the great mathematicians of classical Greece did not stumble on it. Is it that the Greeks had such a marked contempt for applied science, leaving even the instruction of their children to the slaves? But if so, how is it that the nation which gave us geometry and carried this science so far, did not create even a rudimentary algebra? Is it not equally strange that algebra, that cornerstone of modern mathematics, also originated in India and at about the same time when positional numeration did?

A close examination of the anatomy of our modern numeration may shed light on these questions. The principle of position consists in giving the numeral a value which depends not only on the member of the natural sequence it represents, but also on the position it occupies with respect to the other symbols of the group. Thus, the same digit 2 has different meanings in the three numbers 342, 725, 269: in the first case it stands for two; in the second for twenty, in the third for two hundred. As a matter of fact 342 is just an abbreviation for three hundred plus four tens plus two units.

But that is precisely the scheme of the counting board, where 342 is represented by

And, as I said before, it would seem that it is sufficient to translate this scheme into the language of numerals to obtain substantially what we have today.

True! But there is one difficulty. Any attempt to make a permanent record of a counting-board operation would meet the obstacle that such an entry as ≡ = may represent any one of several numbers: 32, 302, 320, 3002, and 3020 among others. In order to avoid this ambiguity it is essential to have some method of representing the gaps, i.e., what is needed is a *symbol for an empty column.*

We see therefore that no progress was possible until a symbol was invented for an *empty* class, a symbol for *nothing,* our modern *zero.* The concrete mind of the ancient Greeks could not conceive the void as a number, let alone endow the void with a symbol.

And neither did the unknown Hindu see in zero the symbol of nothing. The Indian term for zero was *sunya,* which meant *empty* or *blank,* but had no connotation of "void" of "nothing." And so, from all appearances, the discovery of zero was an accident brought about by an attempt to make an unambiguous permanent record of a counting board operation.

How the Indian *sunya* became the zero of today constitutes one of the most interesting chapters in the history of culture. When the Arabs of the tenth century adopted the Indian numeration, they translated the Indian *sunya* by their own, *sifr*, which meant empty in Arabic. When the Indo-Arabic numeration was first introduced into Italy, *sifr* was latinized into *zephirum*. This happened at the beginning of the thirteenth century, and in the course of the next hundred years the word underwent a series of changes which culminated in the Italian *zero*.

About the same time Jordanus Nemararius was introducing the Arabic system into Germany. He kept the Arabic word, changing it slightly to *cifra*. That for some time in the learned circles of Europe the word *cifra* and its derivatives denoted zero is shown by the fact that the great Gauss, the last of the mathematicians of the nineteenth century who wrote in Latin, still used *cifra* in this sense. In the English language the word *cifra* has become *cipher* and has retained its original meaning of zero.

The attitude of the common people toward this new numeration is reflected in the fact that soon after its introduction into Europe, the word *cifra* was used as a secret sign; but this connotation was altogether lost in the succeeding centuries. The verb *decipher* remains as a monument of these early days.

The next stage in this development saw the new art of reckoning spread more widely. It is significant that the essential part played by zero in this new system did not escape the notice of the masses. Indeed, they identified the whole system with its most striking feature, the *cifra*, and this explains how this word in its different forms, *ziffer, chiffre*, etc., came to receive the meaning of numeral, which it has in Europe today.

This double meaning, the popular *cifra* standing for numeral and the *cifra* of the learned signifying zero, caused considerable confusion. In vain did scholars attempt to revive the original meaning of the word: the popular meaning had taken deep root.

The learned had to yield to popular usage, and the matter was eventually settled by adopting the Italian zero in the sense in which it is used today.

The same interest attaches to the word *algorithm*. As the term is used today, it applies to any mathematical procedure consisting of an indefinite number of steps, each step applying to the results of the one preceding it. But between the tenth and fifteenth centuries *algorithm* was synonymous with positional numeration. We now know that the word is merely a corruption of Al Kworesmi, the name of the Arabian mathematician of the ninth century whose book (in Latin translation) was the first work on this subject to reach Western Europe.

Today, when positional numeration has become a part of our daily life, it seems that the superiority of this method, the compactness of its notation, the ease and elegance it introduced in calculations, should have assured the rapid and sweeping acceptance of it. In reality, the transition, far from being immediate, extended over long centuries. The struggle between the *Abacists*, who defended the old traditions, and the *Algorists*, who advocated the reform, lasted from the eleventh to the fifteenth century and went through all the usual stages of obscurantism and reaction. In some places, Arabic numerals were banned from official documents; in others, the art was prohibited altogether. And, as usual, *prohibition* did not succeed in abolishing, but merely served to spread *bootlegging*, ample evidence of which is found in the thirteenth century archives of Italy, where, it appears, merchants were using the Arabic numerals as a sort of secret code.

Yet, for a while reaction succeeded in arresting the progress and in hampering the development of the new system. Indeed, little of essential value or of lasting influence was contributed to the art of reckoning in these transition centuries. Only the outward appearance of the numerals went through a series of changes; not, however, from any desire for improvement, but because the manuals of these days were hand-written. In fact, the numerals did not assume a stable form until the introduction of printing. It can be added parenthetically that so great was the stabilizing influence of printing that the numerals of today have essentially the same appearance as those of the fifteenth century.

As to the final victory of the Algorists, no definite date can be set. We do know that at the beginning of the sixteenth century the supremacy of the new numeration was incontestable. Since then progress was unhampered, so that in the course of the next hundred years all the rules of operations, both on integers and on common and decimal fractions, reached practically the same scope and form in which they are taught today in our schools.

Another century, and the Abacists and all they stood for were so completely forgotten that various peoples of Europe began each to regard the positional numeration as its own national achievement. So, for instance, early in the nineteenth century we find that Arabic numerals were called in Germany *Deutsche* with a view to differentiating them from the *Roman*, which were recognized as of foreign origin.

As to the abacus itself, no traces of it are found in Western Europe during the eighteenth century. Its reappearance early in the nineteenth century occurred under very curious circumstances. The mathematician Poncelet, a general under Napoleon, was captured in the Russian campaign and spent many years in Russia as a prisoner of war. Upon returning to France he brought among other curios, a Russian abacus. For many years

to come, this importation of Poncelet's was regarded as a great curiosity of "barbaric" origin. Such examples of national amnesia abound in the history of culture. How many educated people even today know that only four hundred years ago finger counting was the average man's only means of calculating, while the counting board was accessible only to the professional calculators of the time?

Conceived in all probability as the symbol for an empty column on a counting board, the Indian *sunya* was destined to become the turning-point in a development without which the progress of modern science, industry, or commerce is inconceivable. And the influence of this great discovery was by no means confined to arithmetic. By paving the way to a generalized number concept, it played just as fundamental a rôle in practically every branch of mathematics. In the history of culture the discovery of zero will always stand out as one of the greatest single achievements of the human race.

A great discovery! Yes. But, like so many other early discoveries, which have profoundly affected the life of the race,—not the reward of painstaking research, but a gift from blind chance.