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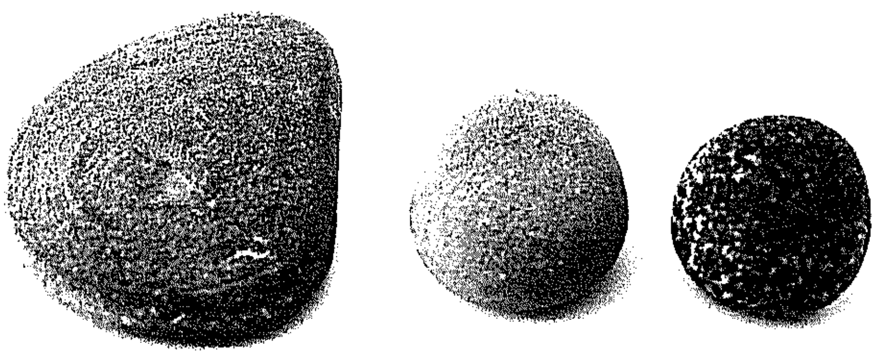
—ALBERT EINSTEIN

number

the language of science



tobias dantzig

edited by joseph mazur | foreword by barry mazur

TOBIAS DANTZIG

NUMBER

The Language of Science

Edited by
JOSEPH MAZUR

Foreword by
BARRY MAZUR

The MASTERPIECE SCIENCE *Edition*

A PLUME BOOK

Can the fundamental issues of the science of number be presented without bringing in the whole intricate apparatus of the science? This book is the author's declaration of faith that it can be done. They who read shall judge!

Tobias Dantzig

Washington, D.C.
May 3, 1930

CHAPTER 1

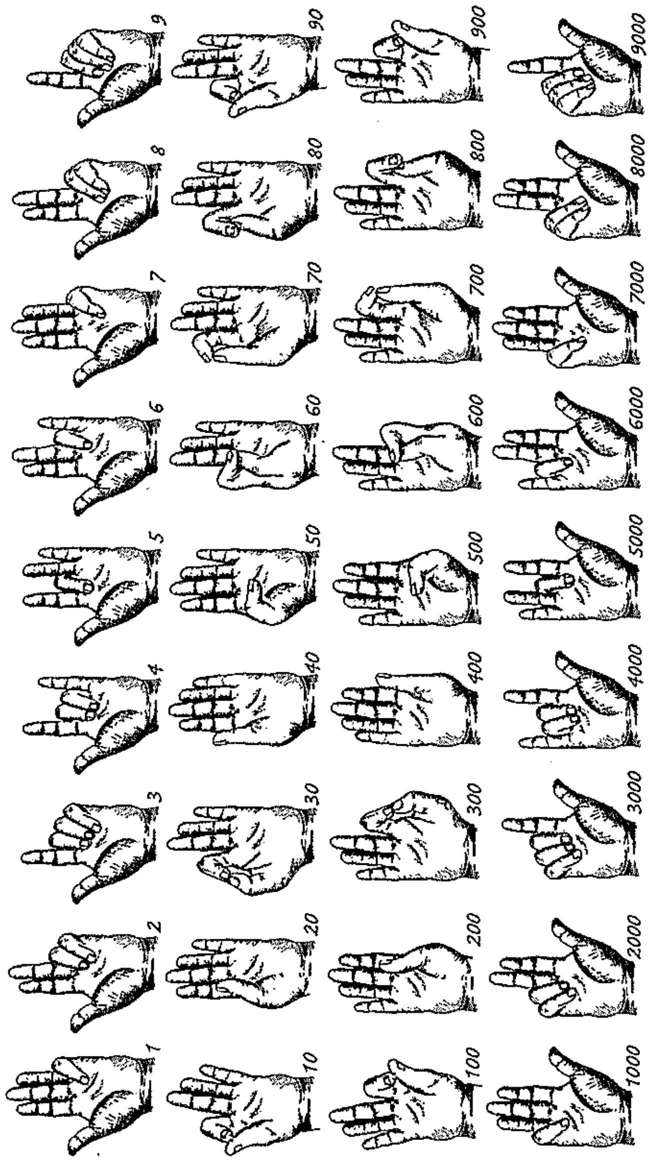
Fingerprints

Ten cycles of the moon the Roman year comprised:
This number then was held in high esteem,
Because, perhaps, on fingers we are wont to count,
Or that a woman in twice five months brings forth,
Or else that numbers wax till ten they reach
And then from one begin their rhythm anew.
—Ovid, *Fasti, III.*

Man, even in the lower stages of development, possesses a faculty which, for want of a better name, I shall call *Number Sense.* This faculty permits him to recognize that something has changed in a small collection when, without his direct knowledge, an object has been removed from or added to the collection.

Number sense should not be confused with counting, which is probably of a much later vintage, and involves, as we shall see, a rather intricate mental process. Counting, so far as we know, is an attribute exclusively human, whereas some brute species seem to possess a rudimentary number sense akin to our own. At least, such is the opinion of competent observers of animal behavior, and the theory is supported by a weighty mass of evidence.

Many birds, for instance, possess such a number sense. If a nest contains four eggs one can safely be taken, but when two are removed the bird generally deserts. In some unaccountable way the bird can distinguish two from three. But this faculty is by no



FINGER SYMBOLS
(FROM A MANUAL PUBLISHED IN 1520)

means confined to birds. In fact the most striking instance we know is that of the insect called the "solitary wasp." The mother wasp lays her eggs in individual cells and provides each egg with a number of live caterpillars on which the young feed when hatched. Now, the number of victims is remarkably constant for a given species of wasp: some species provide 5, others 12, others again as high as 24 caterpillars per cell. But most remarkable is the case of the *Genus Eumenus*, a variety in which the male is much smaller than the female. In some mysterious way the mother knows whether the egg will produce a male or a female grub and apportions the quantity of food accordingly; she does not change the species or size of the prey, but if the egg is male she supplies it with five victims, if female with ten.

The regularity in the action of the wasp and the fact that this action is connected with a fundamental function in the life of the insect make this last case less convincing than the one which follows. Here the action of the bird seems to border on the conscious:

A squire was determined to shoot a crow which made its nest in the watch-tower of his estate. Repeatedly he had tried to surprise the bird, but in vain: at the approach of the man the crow would leave its nest. From a distant tree it would watchfully wait until the man had left the tower and then return to its nest. One day the squire hit upon a ruse: two men entered the tower, one remained within, the other came out and went on. But the bird was not deceived: it kept away until the man within came out. The experiment was repeated on the succeeding days with two, three, then four men, yet without success. Finally, five men were sent: as before, all entered the tower, and one remained while the other four came out and went away. Here the crow lost count. Unable to distinguish between four and five it promptly returned to its nest.

There are primitive languages which have words for every color of the rainbow but have no word for color; there are others which have all number words but no word for number. The same is true of other conceptions. The English language is very rich in native expressions for particular types of collections: *flock*, *herd*, *set*, *lot* and *bunch* apply to special cases; yet the words *collection* and *aggregate* are of foreign extraction.

The concrete preceded the abstract. "It must have required many ages to discover," says Bertrand Russell, "that a brace of pheasants and a couple of days were both instances of the number two." To this day we have quite a few ways of expressing the idea *two*: pair, couple, set, team, twin, brace, etc., etc.

A striking example of the extreme concreteness of the early number concept is the Thimshian language of a British Columbia tribe. There we find seven distinct sets of number words: one for flat objects and animals; one for round objects and time; one for counting men; one for long objects and trees; one for canoes; one for measures; one for counting when no definite object is referred to. The last is probably a later development; the others must be relics of the earliest days when the tribesmen had not yet learned to count.

It is counting that consolidated the concrete and therefore heterogeneous notion of plurality, so characteristic of primitive man, into the *homogeneous abstract number concept*, which made mathematics possible.

Yet, strange though it may seem, it is possible to arrive at a logical, clear-cut number concept without bringing in the artifices of counting.

We enter a hall. Before us are two collections: the seats of the auditorium, and the audience. *Without counting* we can ascertain whether the two collections are equal and, if not equal,

which is the greater. For if every seat is taken and no man is standing, we *know without counting* that the two collections are equal. If every seat is taken and some in the audience are standing, *we know without counting* that there are more people than seats.

We derive this knowledge through a process which dominates all mathematics and which has received the name of *one-to-one correspondence*. It consists in assigning to every object of one collection an object of the other, the process being continued until one of the collections, or both, are exhausted.

The number technique of many primitive peoples is confined to just such such a matching or tallying. They keep the record of their herds and armies by means of notches cut in a tree or pebbles gathered in a pile. That our own ancestors were adept in such methods is evidenced by the etymology of the words *tally* and *calculate*, of which the first comes from the Latin *talea*, cutting, and the second from the Latin *calculus*, pebble.

It would seem at first that the process of correspondence gives only a means for comparing two collections, but is incapable of creating number in the absolute sense of the word. Yet the transition from relative number to absolute is not difficult. It is necessary only to create *model collections*, each typifying a possible collection. Estimating any given collection is then reduced to the selection among the available models of one which can be matched with the given collection member by member.

Primitive man finds such models in his immediate environment: the wings of a bird may symbolize the number two, clover-leaves three, the legs of an animal four, the fingers on his own hand five. Evidence of this origin of number words can be found in many a primitive language. Of course, once the *number word* has been created and adopted, it becomes as good a model as the object it originally represented. The necessity of discriminating between the name of the borrowed object and the number symbol itself would naturally tend to bring about a change in sound, until in the course of time the very connection between the two is lost to memory. As man learns to rely more and more on his language, the sounds supersede the images for which they stood, and the originally concrete models take the abstract form of number words. Memory and habit lend concreteness to these abstract forms, and so mere words become measures of plurality.

The concept I just described is called *cardinal* number. The cardinal number rests on the principle of correspondence: it implies *no counting.* To create a counting process it is not enough to have a motley array of models, comprehensive though this latter may be. We must devise a number *system*: our set of models must be arranged in an ordered sequence, a sequence which progresses in the sense of growing magnitude, the *natural sequence*: one, two, three.... Once this system is created, *counting a collection* means assigning to every member a term in the natural sequence in *ordered succession* until the collection is exhausted. The term of the natural sequence assigned to the *last* member of the collection is called the *ordinal number* of the collection.

The ordinal system may take the concrete form of a rosary, but this, of course, is not essential. The *ordinal* system acquires existence when the first few number words have been committed to memory in their *ordered succession,* and a phonetic scheme has been devised to pass from any larger number to its *successor.*

We have learned to pass with such facility from cardinal to ordinal number that the two aspects appear to us as one. To determine the plurality of a collection, i.e., its cardinal number,

we do not bother any more to find a model collection with which we can match it,—we *count* it. And to the fact that we have learned to identify the two aspects of number is due our progress in mathematics. For whereas in practice we are really interested in the cardinal number, this latter is incapable of creating an arithmetic. The operations of arithmetic are based on the tacit assumption that *we can always pass from any number to its successor,* and this is the essence of the ordinal concept.

And so matching by itself is incapable of creating an art of reckoning. Without our ability to arrange things in ordered succession little progress could have been made. Correspondence and succession, the two principles which permeate all mathematics—nay, all realms of exact thought—are woven into the very fabric of our number system.

It is natural to inquire at this point whether this subtle distinction between cardinal and ordinal number had any part in the early history of the number concept. One is tempted to surmise that the cardinal number, based on matching only, preceded the ordinal number, which requires both matching and ordering. Yet the most careful investigations into primitive culture and philology fail to reveal any such precedence. Wherever any number technique exists at all, both aspects of number are found.

But, also, wherever a counting technique, worthy of the name, exists at all, *finger counting* has been found to either precede it or accompany it. And in his fingers man possesses a device which permits him to pass imperceptibly from cardinal to ordinal number. Should he want to indicate that a certain collection contains four objects he will raise or turn down four fingers *simultaneously;* should he want to count the same collection he will raise or turn down these fingers *in succession.* In the first case he is using his fingers as a cardinal model, in the second as an ordinal system. Unmistakable traces of this origin of counting are found in practically every primitive language. In most of these tongues the number "five" is expressed by "hand," the number "ten" by "two hands," or sometimes by "man." Furthermore, in many primitive languages the number-words up to four are identical with the names given to the four fingers.

The more civilized languages underwent a process of attrition which obliterated the original meaning of the words. Yet here too "fingerprints" are not lacking. Compare the Sanskrit *pantcha*, five, with the related Persian *pentcha*, hand; the Russian "piat," five, with "piast," the outstretched hand.

It is to his *articulate ten fingers* that man owes his success in calculation. It is these fingers which have taught him to count and thus extend the scope of number indefinitely. Without this device the number technique of man could not have advanced far beyond the rudimentary number sense. And it is reasonable to conjecture that without our fingers the development of number, and consequently that of the exact sciences, to which we owe our material and intellectual progress, would have been hopelessly dwarfed.

And yet, except that our children still learn to count on their fingers and that we ourselves sometimes resort to it as a gesture of emphasis, finger counting is a lost art among modern civilized people. The advent of writing, simplified numeration, and universal schooling have rendered the art obsolete and superfluous. Under the circumstances it is only natural for us to underestimate the rôle that finger counting has played in the history of reckoning. Only a few hundred years ago finger counting was such a widespread custom in Western Europe that no manual of arithmetic was complete unless it gave full instructions in the method. (See page 2.)

The art of using his fingers in counting and in performing the simple operations of arithmetic, was then one of the accomplishments of an educated man. The greatest ingenuity was displayed in devising rules for adding and multiplying numbers on one's fingers. Thus, to this day, the peasant of central France (Auvergne) uses a curious method for multiplying numbers above 5. If he wishes to multiply 9×8, he bends down 4 fingers on his left hand (4 being the excess of 9 over 5), and 3 fingers on his right hand ($8 - 5 = 3$). Then the number of the bent-down fingers gives him the tens of the result ($4 + 3 = 7$), while the product of the unbent fingers gives him the units ($1 \times 2 = 2$).

Artifices of the same nature have been observed in widely separated places, such as Bessarabia, Serbia and Syria. Their striking similarity and the fact that these countries were all at one time parts of the great Roman Empire, lead one to suspect the Roman origin of these devices. Yet, it may be maintained with equal plausibility that these methods evolved independently, similar conditions bringing about similar results.

Even today the greater portion of humanity is counting on fingers: to primitive man, we must remember, this is the only means of performing the simple calculations of his daily life.

How old is our number language? It is impossible to indicate the exact period in which number words originated, yet there is unmistakable evidence that it preceded written history by many thousands of years. One fact we have mentioned already: all traces of the original meaning of the number words in European languages, with the possible exception of *five*, are lost. And this is the more remarkable, since, as a rule, number words possess an extraordinary stability. While time has wrought radical changes in all other aspects we find that the number vocabulary has been practically unaffected. In fact this stability is utilized by philologists to trace kinships between apparently remote language groups. The reader is invited to examine the table at the end of the chapter where the number words of the standard Indo-European languages are compared.

Why is it then that in spite of this stability no trace of the original meaning is found? A plausible conjecture is that while number words have remained unchanged since the days when they originated, the names of the concrete objects from which the number words were borrowed have undergone a complete metamorphosis.

As to the structure of the number language, philological researches disclose an almost universal uniformity. Everywhere the ten fingers of man have left their permanent imprint.

Indeed, there is no mistaking the influence of our ten fingers on the "selection" of the base of our number system. In all Indo-European languages, as well as Semitic, Mongolian, and most primitive languages, the base of numeration is ten, i.e., there are independent number words up to ten, beyond which some compounding principle is used until 100 is reached. All these languages have independent words for 100 and 1000, and some languages for even higher decimal units. There are apparent exceptions, such as the English *eleven* and *twelve*, or the German *elf* and *zwölf*, but these have been traced to *ein-lif* and *zwo-lif; lif* being old German for *ten.*

It is true that in addition to the decimal system, two other bases are reasonably widespread, but their character confirms to a remarkable degree the *anthropomorphic* nature of our counting scheme. These two other systems are the quinary, base 5, and the vigesimal, base 20.

In the *quinary* system there are independent number words up to *five,* and the compounding begins thereafter. (See table at the end of chapter.) It evidently originated among people who had the habit of counting on one hand. But why should man confine himself to one hand? A plausible explanation is that primitive man rarely goes about unarmed. If he wants to count, he tucks his weapon under his arm, the left arm as a rule, and counts on his left hand, using his right hand as check-off. This may explain why the left hand is almost universally used by right-handed people for counting.

Many languages still bear the traces of a quinary system, and it is reasonable to believe that some decimal systems passed through the quinary stage. Some philologists claim that even the Indo-European number languages are of a quinary origin. They point to the Greek word *pempazein,* to count by fives, and also to the unquestionably quinary character of the Roman numerals. However, there is no other evidence of this sort, and it is much more probable that our group of languages passed through a preliminary *vigesimal stage.*

This latter probably originated among the primitive tribes who counted on their toes as well as on their fingers. A most striking example of such a system is that used by the Maya Indians of Central America. Of the same general character was the system of the ancient Aztecs. The day of the Aztecs was divided into 20 hours; a division of the army contained 8000 soldiers ($8000 = 20 \times 20 \times 20$).

While pure vigesimal systems are rare, there are numerous languages where the decimal and the vigesimal systems have merged. We have the English *score, two-score,* and *three-score;* the *French vingt* (20) and *quatre-vingt* (4×20). The old French used this form still more frequently; a hospital in Paris originally built for 300 blind veterans bears the quaint name of

Quinze-Vingt (Fifteen-score); the name *Onze-Vingt* (Eleven-score) was given to a corps of police-sergeants comprising 220 men.

There exists among the most primitive tribes of Australia and Africa a system of numeration which has neither 5, 10, nor 20 for base. It is a *binary* system, i.e., of base two. These savages have not yet reached finger counting. They have independent numbers for one and two, and composite numbers up to six. Beyond six everything is denoted by "heap."

Curr, whom we have already quoted in connection with the Australian tribes, claims that most of these count by pairs. So strong, indeed, is this habit of the native that he will rarely notice that two pins have been removed from a row of seven; he will, however, become immediately aware if one pin is missing. His sense of *parity* is stronger than his number sense.

Curiously enough, this most primitive of bases had an eminent advocate in relatively recent times in no less a person than Leibnitz. A binary numeration requires but two symbols, 0 and 1, by means of which all other numbers are expressed, as shown in the following table:

Decimal	1	2	3	4	5	6	7	8
Binary	1	10	11	100	101	110	111	1000

Decimal	9	10	11	12	13	14	15	16
Binary	1001	1010	1011	1100	1101	1110	1111	10000

The advantages of the *base two* are economy of symbols and tremendous simplicity in operations. It must be remembered that every system requires that tables of addition and multiplication be

committed to memory. For the binary system these reduce to $1 + 1 = 10$ and $1 \times 1 = 1$; whereas for the decimal, each table has 100 entries. Yet this advantage is more than offset by lack of compactness: thus the decimal number $4096 = 2^{12}$ would be expressed in the binary system by 1,000,000,000,000.

It is the mystic elegance of the binary system that made Leibnitz exclaim: *Omnibus ex nihil ducendis sufficit unum.* (One suffices to derive all out of nothing.) Says Laplace:

> "Leibnitz saw in his binary arithmetic the image of Creation ... He imagined that Unity represented God, and Zero the void; that the Supreme Being drew all beings from the void, just as unity and zero express all numbers in his system of numeration. This conception was so pleasing to Leibnitz that he communicated it to the Jesuit, Grimaldi, president of the Chinese tribunal for mathematics, in the hope that this emblem of creation would convert the Emperor of China, who was very fond of the sciences. I mention this merely to show how the prejudices of childhood may cloud the vision even of the greatest men!"

It is interesting to speculate what turn the history of culture would have taken if instead of flexible fingers man had had just two "inarticulate" stumps. If any system of numeration could at all have developed under such circumstances, it would have probably been of the binary type.

That mankind adopted the decimal system is a *physiological accident.* Those who see the hand of Providence in everything will have to admit that Providence is a poor mathematician. For outside its physiological merit the decimal base has little to commend itself. Almost any other base, with the possible exception of *nine,* would have done as well and probably better.

Indeed, if the choice of a base were left to a group of experts, we should probably witness a conflict between the practical man, who would insist on a base with the greatest number of divisors, such as *twelve,* and the mathematician, who would want a prime number, such as *seven* or *eleven,* for a base. As a matter of fact, late in the eighteenth century the great naturalist Buffon proposed that the duodecimal system (base 12) be universally adopted. He pointed to the fact that 12 has 4 divisors, while 10 has only two, and maintained that throughout the ages this inadequacy of our decimal system had been so keenly felt that, in spite of ten being the universal base, most measures had 12 secondary units.

On the other hand the great mathematician Lagrange claimed that a prime base is far more advantageous. He pointed to the fact that with a prime base every systematic fraction would be irreducible and would therefore represent the number in a unique way. In our present numeration, for instance, the decimal fraction .36 stands really for many fractions: 36/100, 18/50, and 9/25 .... Such an ambiguity would be considered lessened if a prime base, such as eleven, were adopted.

But whether the enlightened group to whom we would entrust the selection of the base decided on a prime or a composite base, we may rest assured that the number *ten* would not even be considered, for it is neither prime nor has it a sufficient number of divisors.

In our own age, when calculating devices have largely supplanted mental arithmetic, nobody would take either proposal seriously. The advantages gained are so slight, and the tradition of counting by tens so firm, that the challenge seems ridiculous.

From the standpoint of the history of culture a change of base, even if practicable, would be highly undesirable. As long as

man counts by tens, his ten fingers will remind him of the human origin of this most important phase of his mental life. So may the decimal system stand as a living monument to the proposition:

Man is the measure of all things.